N0. Theoretical Physics Continues (Part III).

Augustin-Jean Fresnel (1788-1827)
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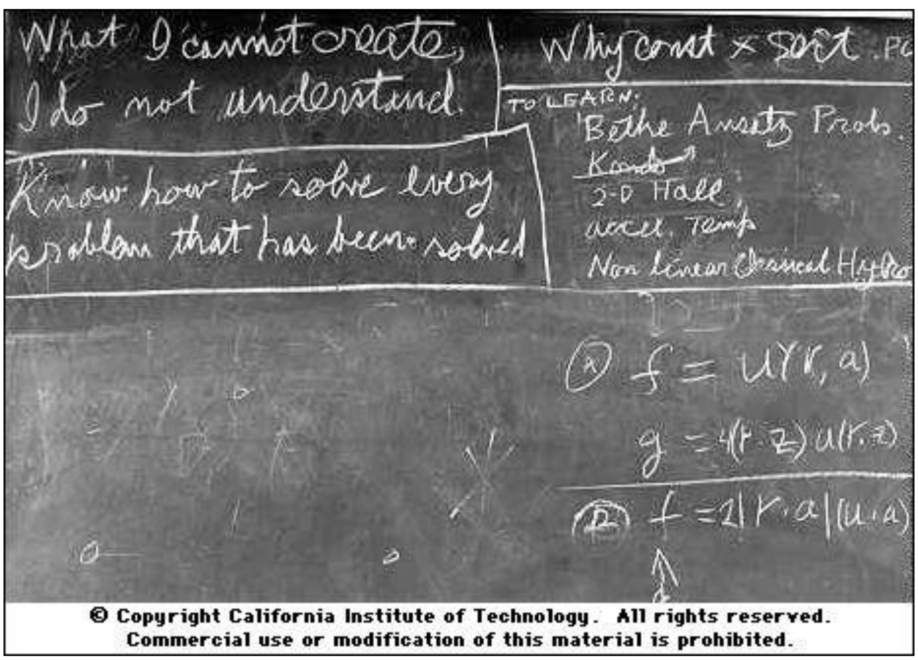
Civil Engineer and Physicist

We have finished a week dedicated to theoretical physics, but there is more. Our main scientist for today is Fresnel, an excellent example of a theoretical physicist. But he was also an outstanding engineer. Remember his invention of the Fresnel lens for light houses?

Before we start, let's reflect on the philosophy of the derivation and address the following question: **Why Derive Everything?**

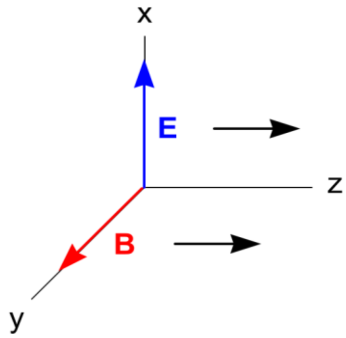
I have always felt it important to derive everything from scratch as far back as I was a student. I was pleased to learn that Feynman had a similar attitude. See his blackboard below at the time of his death. The philosophy is stated succinctly" "What I cannot create, I do not understand."



Richard Feynman
1918 - 1988

If Feynman couldn't derive it, he felt he did not understand it. Underneath this statement on his blackboard appears: "Know how to solve every problem that has been solved." That statement is very ambitious. However, in our course we follow the goal of deriving everything, including some needed trig identities along the way.

N1. The Plane-Wave Model.



$$E_x = E_o \sin(kz - \omega t)$$
$$B_y = B_o \sin(kz - \omega t)$$
$$\frac{E_o}{B_o} = c = \frac{\omega}{k} \qquad c = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$$

Simple Vector form: $\vec{E} = E_o \sin(kz - \omega t)\hat{i}$ and $\vec{B} = B_o \sin(kz - \omega t)\hat{j}$.

Focus on the electric field since our **B** tags along with the rule $\hat{i} \times \hat{j} = \hat{k}$,

where is $\hat{k}$ the propagation direction.

Note that i is used as $i = \sqrt{-1}$ and differently in the unit vector $\hat{i}$.

Likewise k is used in $c = \dfrac{\omega}{k}$ and differently in the unit vector $\hat{k}$.

In general we often write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Let's promote the k in $kz - \omega t$ to a vector pointing in the propagation direction: $\vec{k} = k\hat{k}$.

You have to keep the three k appearances straight:

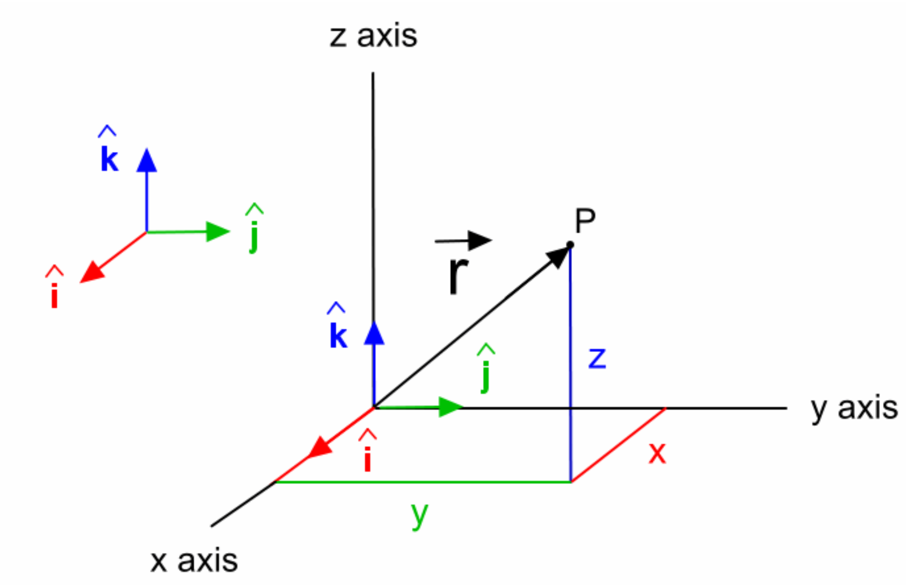
- the vector $\vec{k}$,
- the magnitude k of the vector $\vec{k}$,
- the unit vector $\hat{k}$ that points in the z-direction for our case.

Note that $\vec{k} \cdot \vec{r} = k\hat{k} \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = kx\hat{k} \cdot \hat{i} + ky\hat{k} \cdot \hat{j} + kz\hat{k} \cdot \hat{k} = kz$

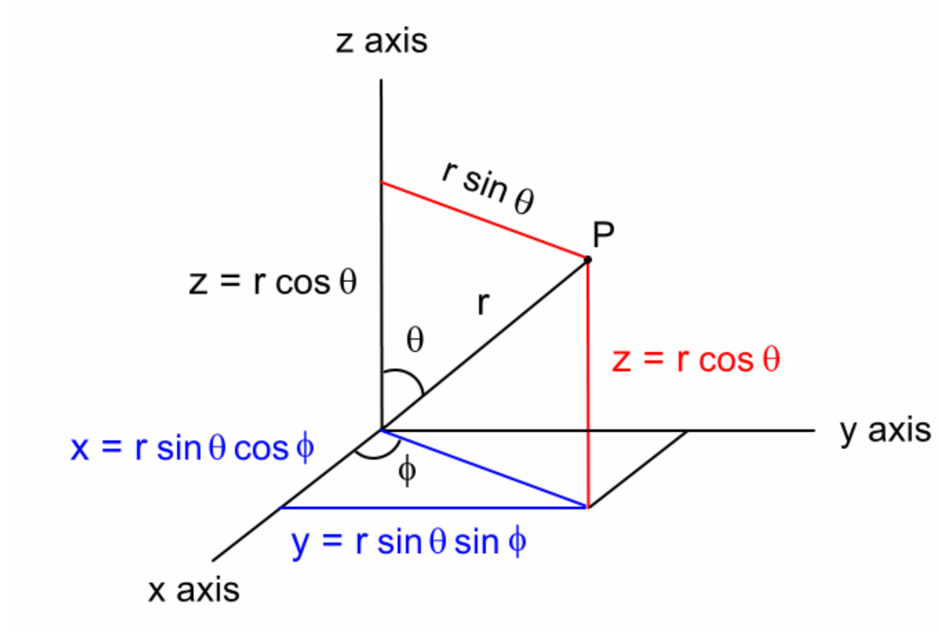
since $\hat{k} \cdot \hat{i} = 0$, $\hat{k} \cdot \hat{j} = 0$, and $\hat{k} \cdot \hat{k} = 1$. Note that $\vec{k} \cdot \vec{r} = kz$,

we can write $\vec{E} = E_o \sin(kz - \omega t)\hat{i}$ as $\vec{E} = E_o \sin(\vec{k} \cdot \vec{r} - \omega t)\hat{i}$.

Since we are reviewing unit vector dot products, let's review $\vec{r}$.



The vector $\vec{r}$ is intimately connected to spherical coordinates.

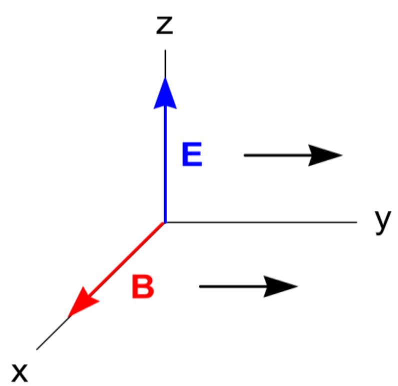


$$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

In the x-y plane you have $\theta = 90^\circ$ and the usual polar coordinates with

(r,ϕ), $x = r\cos\phi$, and $y = r\sin\phi$.

To adapt our wave to the usual labeling of coordinates, we make a minor adjustment.



$\vec{E} = E_o \sin(ky - \omega t)\hat{k}$ and $\vec{B} = B_o \sin(ky - \omega t)\hat{i}$.

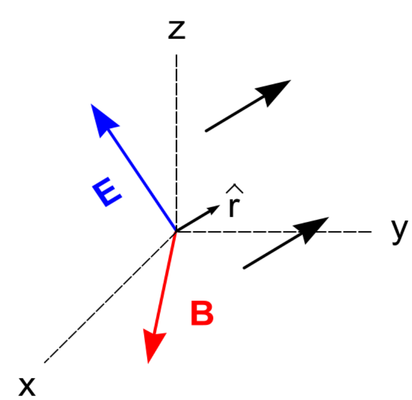
Note that the vector directions for the electric and magnetic fields are still related by

$\hat{E} \times \hat{B}$ => the propagation direction $\hat{n}$,

which is $\hat{j}$ in this case.

Now $\vec{k} = k\hat{j}$ and we can write $\vec{E} = E_o \sin(\vec{k} \cdot \vec{r} - \omega t)\hat{k}$.

What about an arbitrary direction?



We simply let $\vec{k} = k\hat{r}$ and obtain

$\vec{k} \cdot \vec{r} - \omega t = kr - \omega t$ for the sine argument.

We can leave the general $\vec{k} \cdot \vec{r} - \omega t$ form. However, what do we do for the unit vector in the direction of $\vec{E}$? Of course that $\hat{k}$ has to go. Since the electric and magnetic field combo can in general be in any perpendicular direction to the propagation direction, we promote E_o to a vector so that the amplitude can handle it.

$$\vec{E} = \vec{E_o} \sin(\vec{k} \cdot \vec{r} - \omega t) ,$$

The direction for $\vec{E_o}$, or $\vec{E}$, is the polarization direction or simply **polarization**.

And we can include an additional arbitrary phase ϕ .

$$\vec{E} = \vec{E_o} \sin(\vec{k} \cdot \vec{r} - \omega t + \phi)$$

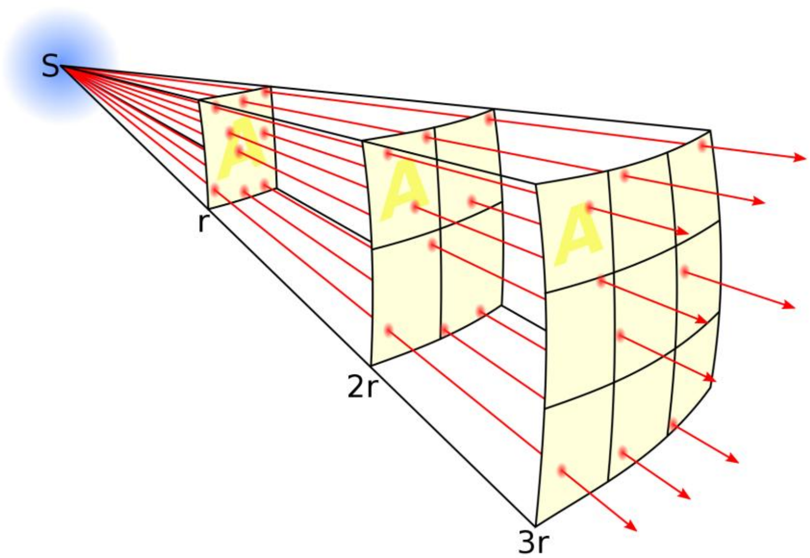
The final step is to use the exponential representation as exponentials are much easier to work with. Remember before when we calculated an intensity? We took

$I \sim \left|\vec{E}\right|^2$ with the complex conjugate trick and the imaginary pieces went away when we analyzed the two-slit interference problem. A great advantage we saw there was that we could add the complex wave forms from each slit and that took care of all the phases with their associated interferences. We also did that two-slit interference problem the regular way without phasors. The same result was obtained in each case, leading us to trust the exponential protocol even though it is a bit more mathematically abstruse. So we continue trusting and write for a general wave

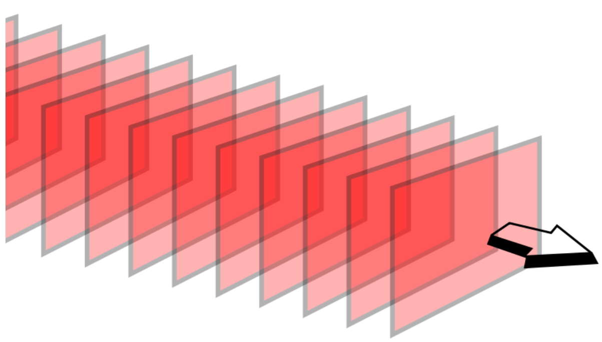
$$\vec{E} = \vec{E_o} e^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)} \text{ or } \vec{E} = \vec{E_o} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \text{ if we can set } \phi = 0 .$$

We will use the form $\vec{E} = \vec{E_o} \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$ often so that parameters are easier to see.

Many consider the electric field along the path of the wave as a cute little patch in the plane perpendicular to the direction of propagation, inspired by the cute little patches of the wave front you see below. Such an association leads to our wave being described as a **plane wave**.



Wavefronts as Cute Little Plane Patches. Wikipedia: Borb. Creative Commons License



Plane Wave Traveling in 3D. Wikimedia: Fffred and Quibik. Released to Public Domain

N2. Maxwell Equations in Matter. We have to be careful because now we need the Maxwell equations in matter. We replace ε_o with ε and μ_o with μ. And to emphasize free charges and currents, we add the subscript *free*. The Maxwell equations in materials are shown below along with the vacuum equations.

Maxwell Equations In Vacuum	Maxwell Equations In Linear and Isotropic Matter
$\oiint \vec{E} \cdot \overrightarrow{dA} = \dfrac{Q}{\varepsilon_0}$	$\oiint \vec{E} \cdot \overrightarrow{dA} = \dfrac{Q_{free}}{\varepsilon}$
$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$	$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$
$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \dfrac{d\Phi_E}{dt}$	$\oint \vec{B} \cdot \overrightarrow{dl} = \mu i_{free} + \mu\varepsilon \dfrac{d\Phi_E}{dt}$
$\oint \vec{E} \cdot \overrightarrow{dl} = -\dfrac{d\Phi_B}{dt}$	$\oint \vec{E} \cdot \overrightarrow{dl} = -\dfrac{d\Phi_B}{dt}$

The medium's permittivity $\varepsilon = \varepsilon_o(1+\chi_e)$, where χ_e is the electric susceptibility.

The medium's permeability $\mu = \mu_o(1+\chi_m)$, where χ_m is the magnetic susceptibility.

The electric susceptibility is the material's susceptibility of displacing bound charges in response to an applied electric field to the substance. Only the free charges appear explicitly in the first Maxwell equation. Note the free is added as a subscript to emphasize this fact. We also include free as a subscript for the current to emphasize that freely moving charges produce the current. The magnetic susceptibility is the material's susceptibility of becoming magnetized when a magnetic field is applied to the substance.

But we are in luck since we are only interested in nonmagnetic materials like glass. For such nonmagnetic materials, χ_m is essentially zero and we can take $\mu = \mu_o$. Our chosen materials such as glass are not good conductors of electricity. Therefore, $i_{free} = 0$. The two Maxwell equations we will need are the last two. When set up for nonmagnetic substances, the third Maxwell equation is the only one we need to adapt, as shown below.

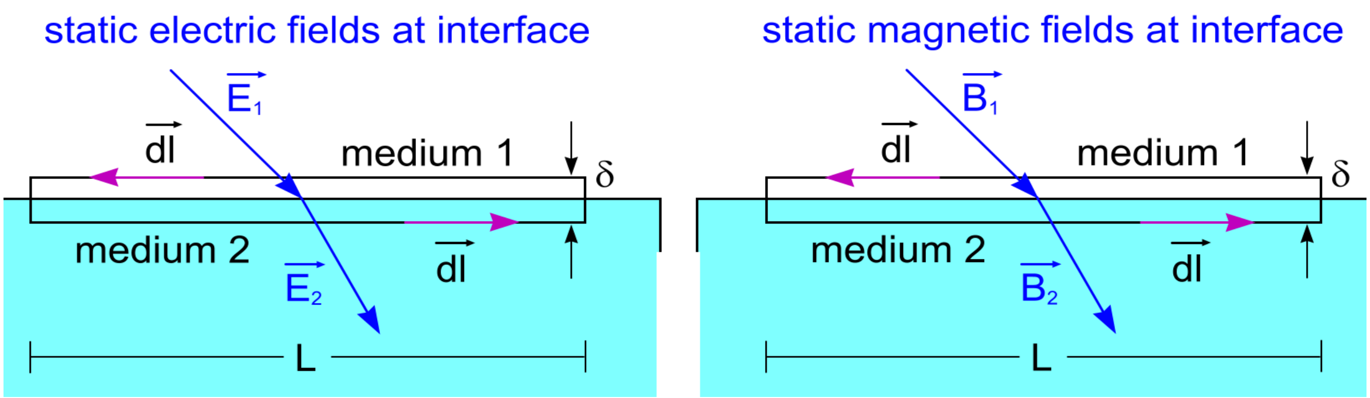
$$\oint \vec{B} \cdot \vec{dl} = \mu_o \varepsilon \frac{d\Phi_E}{dt} \quad \text{and} \quad \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} .$$

Note that we expect ε for the glass to be different from vacuum because if $\mu = \mu_o$, we need $\varepsilon \neq \varepsilon_o$ to get a different speed for light in glass: $v = \dfrac{1}{\sqrt{\varepsilon \mu_o}}$. The index of refraction gives

$$n \equiv \frac{c}{v} = \frac{\dfrac{1}{\sqrt{\varepsilon_o \mu_o}}}{\dfrac{1}{\sqrt{\varepsilon \mu_o}}} \quad \text{and} \quad \boxed{n = \sqrt{\frac{\varepsilon}{\varepsilon_o}}}.$$

N3. Interface Boundary Conditions. We start with

$$\oint \vec{B} \cdot \vec{dl} = \mu_o \varepsilon \frac{d\Phi_E}{dt} \quad \text{and} \quad \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} .$$



At the interface we make two rectangular path loops. We make the loop thickness a small δ. The area $L\delta$ is so small that the fluxes are vanishingly small. Then,

$$\oint \vec{B} \cdot \vec{dl} = 0 \quad \text{and} \quad \oint \vec{E} \cdot \vec{dl} = 0, \text{ simple enough!}$$

Consider $\oint \vec{E} \cdot \vec{dl} = E_{2||}L - E_{1||}L = (E_{2||} - E_{1||})L = 0$, where the subscript || means parallel or tangent component. We obtain the following boundary conditions:

$$E_{1||} = E_{2||} \quad \text{and} \quad B_{1||} = B_{2||} .$$

N4. An EM Wave at an Interface. What happens when an EM wave hits an interface such as an air-glass interface? We set the problem up by considering an incident wave, reflected wave, and transmitted wave. We choose $\phi = 0$ for all the waves. Such a choice means that the incoming wave will have $\phi = 0$, but if relative phases should enter, they will do so via the phasors for the reflected and transmitted beams.

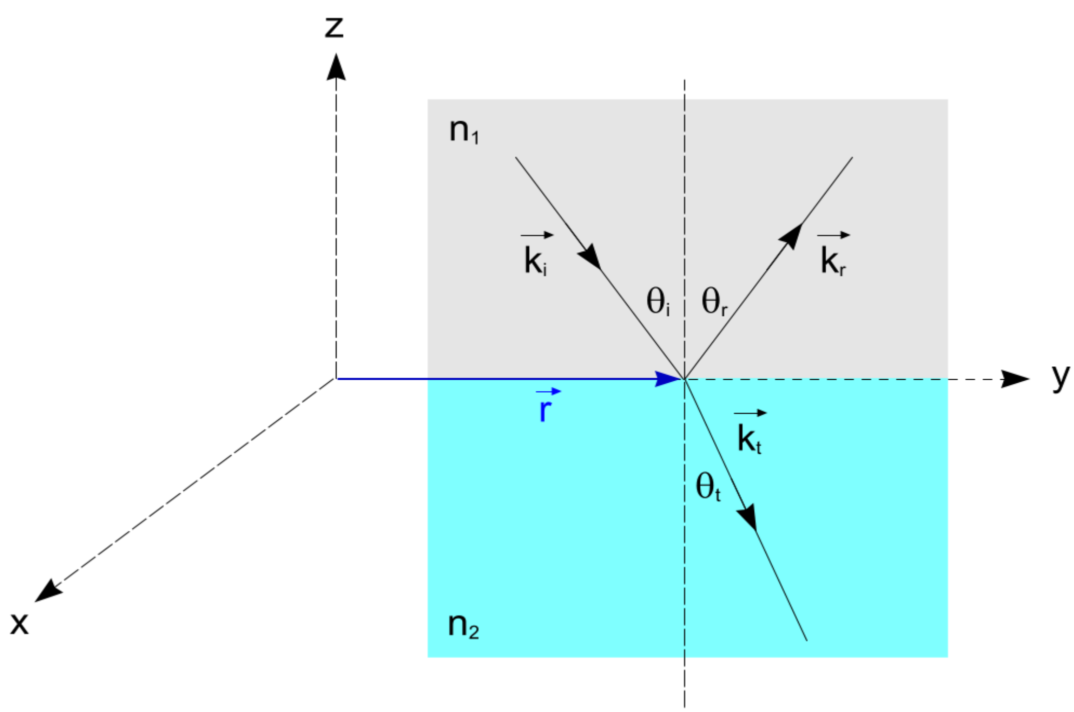
$$\vec{E}_i = \vec{E}_{io} \exp[i(\vec{k_i} \cdot \vec{r} - \omega_i t)]$$
$$\vec{E}_r = \vec{E}_{ro} \exp[i(\vec{k_r} \cdot \vec{r} - \omega_r t)]$$
$$\vec{E}_t = \vec{E}_{to} \exp[i(\vec{k_t} \cdot \vec{r} - \omega_t t)]$$

In the figure the incident light is from above. Then the general solution is

$$\vec{E}_{io} \exp[i(\vec{k_i} \cdot \vec{r} - \omega_i t)] + \vec{E}_{ro} \exp[i(\vec{k_r} \cdot \vec{r} - \omega_r t)] \text{ for } z > 0,$$

$$\vec{E}_t = \vec{E}_{to} \exp[i(\vec{k_t} \cdot \vec{r} - \omega_t t)] \text{ for } z < 0.$$

There is no incident wave entering from below per our initial set-up conditions.



When we apply the boundary condition $E_{1\|} = E_{2\|}$.

$$\{\vec{E}_{io} \exp[i(\vec{k}_i \cdot \vec{r} - \omega_i t)] + \vec{E}_{ro} \exp[i(\vec{k}_r \cdot \vec{r} - \omega_r t)]\}_{\|} = \{\vec{E}_{to} \exp[i(\vec{k}_t \cdot \vec{r} - \omega_t t)]\}_{\|}$$

The tangential component refers to the vector part and not the phasor.

$$\vec{E}_{io\|} \exp[i(\vec{k}_i \cdot \vec{r} - \omega_i t)] + \vec{E}_{ro\|} \exp[i(\vec{k}_r \cdot \vec{r} - \omega_r t)] = \vec{E}_{to\|} \exp[i(\vec{k}_t \cdot \vec{r} - \omega_t t)]$$

This equation must always be true for all the different $(\vec{r}, t)$.

(a) True for All Times. Due to the arbitrariness in choosing time t, we must have for all t

$\omega_i = \omega_r = \omega_t$, i.e., the frequency does not change.

Remember that $\omega = kv$, $k = \dfrac{2\pi}{\lambda}$, and $n = \dfrac{c}{v}$. From $\omega = kv$,

$$k_i v_i = k_r v_r = k_t v_t \ .$$

Using $v = \dfrac{c}{n}$ leads to $k_i \dfrac{c}{n_1} = k_r \dfrac{c}{n_1} = k_t \dfrac{c}{n_2}$.

Substituting $k = \dfrac{2\pi}{\lambda}$ gives us $\dfrac{2\pi}{\lambda_i} \dfrac{c}{n_1} = \dfrac{2\pi}{\lambda_r} \dfrac{c}{n_1} = \dfrac{2\pi}{\lambda_t} \dfrac{c}{n_2}$.

Cancelling the constants, $\dfrac{1}{\lambda_i} \dfrac{1}{n_1} = \dfrac{1}{\lambda_r} \dfrac{1}{n_1} = \dfrac{1}{\lambda_t} \dfrac{1}{n_2}$ and

$$\lambda_i n_1 = \lambda_r n_1 = \lambda_t n_2 \ .$$

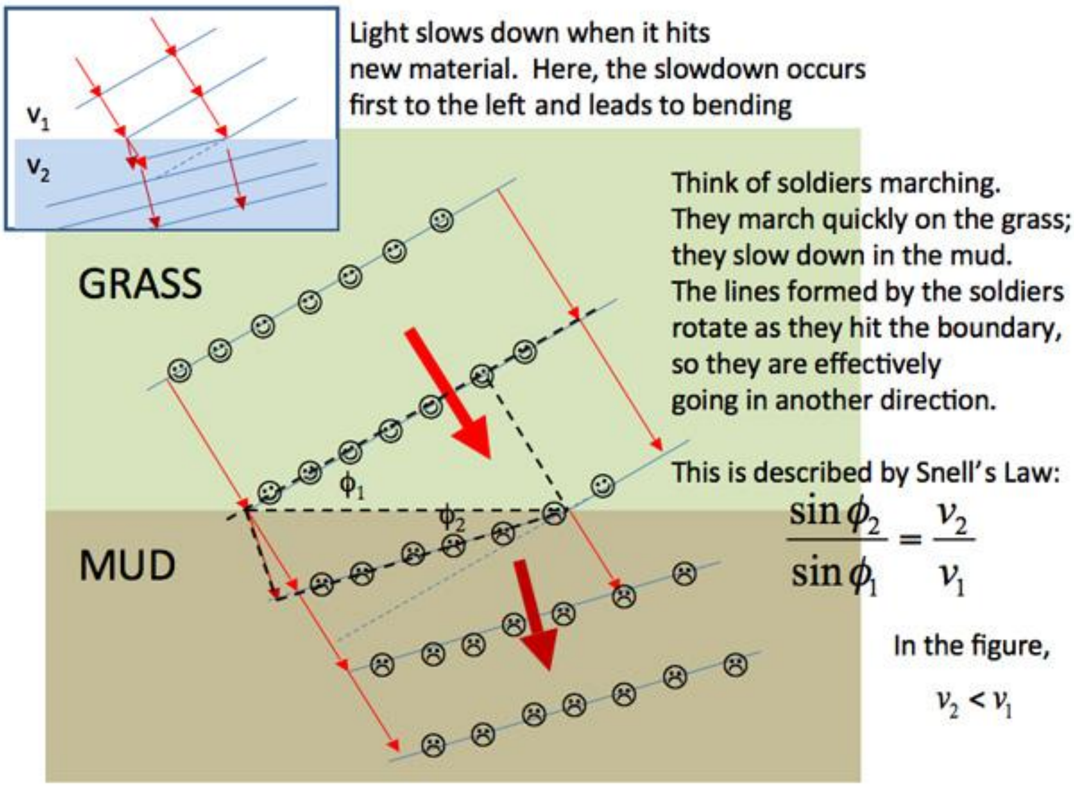
The wavelength of the reflected wave equals the incident wavelength: $\lambda_r = \lambda_i$.

The wavelength of the transmitted wave is $\lambda_t = \dfrac{n_1}{n_2} \lambda_i$.

If $n_1 < n_2$, then the wavelength shortens, i.e., $\lambda_t < \lambda_i$.

Remember observing this phenomena way back with the marching soldiers model?

Can you believe that the Maxwell Equations are teaching us all this from first principles?



Refraction and Marching Soldiers. Image Courtesy University Corporation for Atmospheric Research (NCAR), Boulder, Colorado, Material Supported by the National Science Foundation (NSF) and NCAR.

Two observations we made earlier are derived here:

1. Conservation of soldiers – the frequency of passing columns remains the same.

2. Shorter Wavelength – the wavelength must shorter if the soldiers slow their pace.

All because of the Maxwell equations.

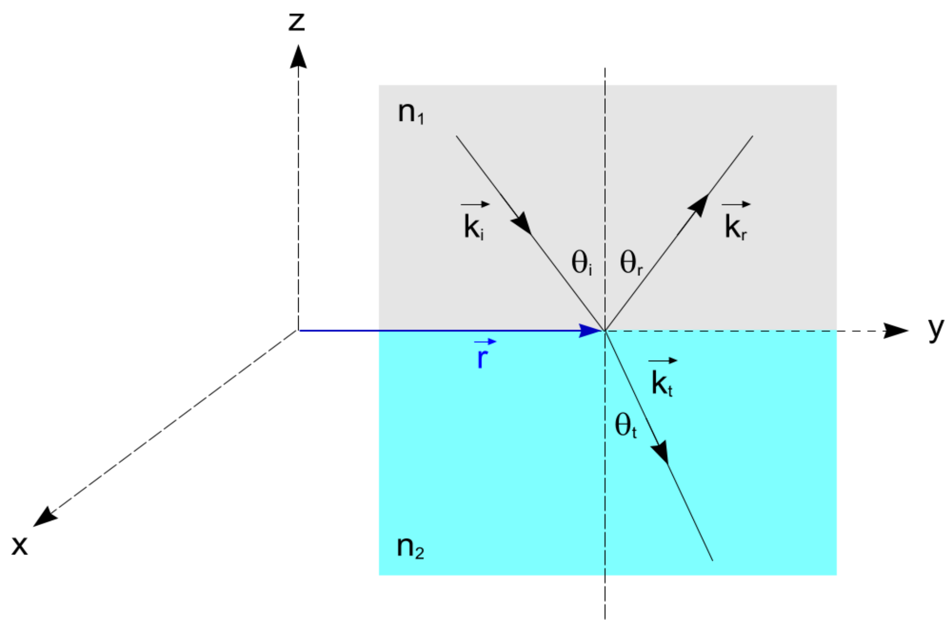
Another Big Surprise to Come Shortly!

(b) True for All Points Along the Interface. Due to the arbitrariness in choosing a specific point along the interface, in

$$\vec{E}_{io\|}\exp[i(\vec{k}_i\cdot\vec{r}-\omega_i t)]+\vec{E}_{ro\|}\exp[i(\vec{k}_r\cdot\vec{r}-\omega_r t)]=\vec{E}_{to\|}\exp[i(\vec{k}_t\cdot\vec{r}-\omega_t t)],$$

we must have $\vec{k}_i\cdot\vec{r}=\vec{k}_r\cdot\vec{r}=\vec{k}_t\cdot\vec{r}$ at the interface.

Since $\vec{k}_i\cdot\vec{r}=\vec{k}_r\cdot\vec{r}=\vec{k}_t\cdot\vec{r}$ anywhere along the interface, pick $\vec{r}=r\,\hat{j}$ in the figure.



We obtain $k_i r\cos(90°-\theta_i)=k_r r\cos(90°-\theta_r)=k_t r\cos(90°-\theta_t)$.

$$k_i\sin\theta_i=k_r\sin\theta_r=k_t\sin\theta_t$$

We recall $k=\dfrac{2\pi}{\lambda}$, $v=\lambda f$, and $n=\dfrac{c}{v}$.

Then $k=\dfrac{2\pi}{\lambda}=2\pi\dfrac{f}{v}=2\pi f\dfrac{n}{c}=\omega\dfrac{n}{c}$ and

$k_i\sin\theta_i=k_r\sin\theta_r=k_t\sin\theta_t$ becomes

$$\omega_i \frac{n_1}{c} \sin\theta_i = \omega_r \frac{n_1}{c} \sin\theta_r = \omega_t \frac{n_2}{c} \sin\theta_t .$$

But $\omega_i = \omega_r = \omega_t$, i.e., the frequencies are all the same ("conserving soldiers").

Then $n_1 \sin\theta_i = n_1 \sin\theta_r = n_2 \sin\theta_t$.

What?

The Law of Reflection and the Law of Refraction from Maxwell's Equations?

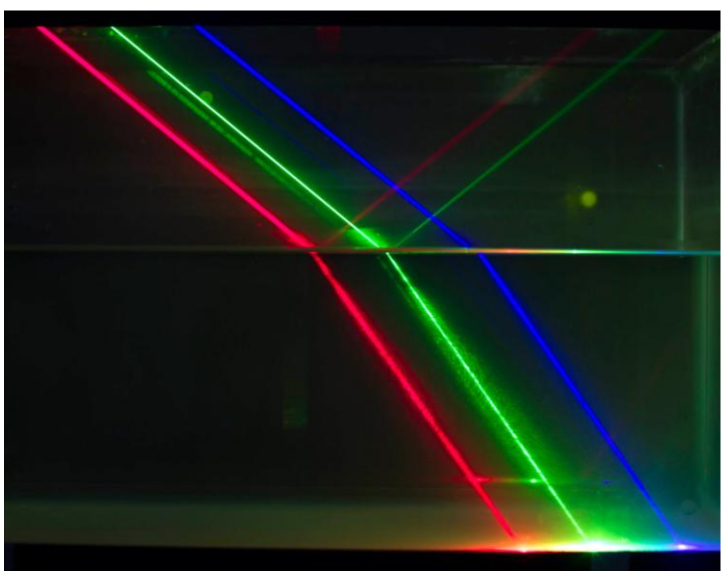
(a) **Law of Reflection:** $n_1 \sin\theta_i = n_1 \sin\theta_r$ => $\sin\theta_i = \sin\theta_r$ =>

$$\boxed{\theta_i = \theta_r}$$

(b) **Law of Refraction:** $n_1 \sin\theta_i = n_2 \sin\theta_t$ =>

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

And there's even more coming from the Maxwell Equations thanks to Fresnel!



Courtesy Pascals-Puppy Blogspot (2011)

N6. The Fresnel Equations. Fresnel, the best of both engineering and theoretical physics, died at the young age of 39 due to tuberculosis. You will shortly see his beautiful wave treatment of optics as we further explore reflection and refraction of electromagnetic waves at a boundary of two media. A poet and a musician whose lives overlapped with Fresnel also died young of tuberculosis. These three artists in their own right are shown below.

A physicist-engineer, poet, and a pianist-composer all dying of tuberculosis at a young age.

Physicist-Engineer	Poet	Pianist-Composer
Augustin-Jean Fresnel	John Keats	Frédéric Chopin
		
1788 - 1827	1795 - 1821	1810 - 1849
Died at Age 39	Died at Age 25	Died at Age 39
Early Wave Optics Period	Romantic Period in Literature	Romantic Period in Music
Fresnel Equations	*Ode to a Nightingale*	*Third Scherzo*

Keats *Ode to a Nightingale* 1819 Last Stanza	Forlorn! the very word is like a bell To toll me back from thee to my sole self! Adieu! the fancy cannot cheat so well As she is fam'd to do, deceiving elf. Adieu! adieu! thy plaintive anthem fades Past the near meadows, over the still stream, Up the hill-side; and now 'tis buried deep In the next valley-glades: Was it a vision, or a waking dream? Fled is that music:—Do I wake or sleep?

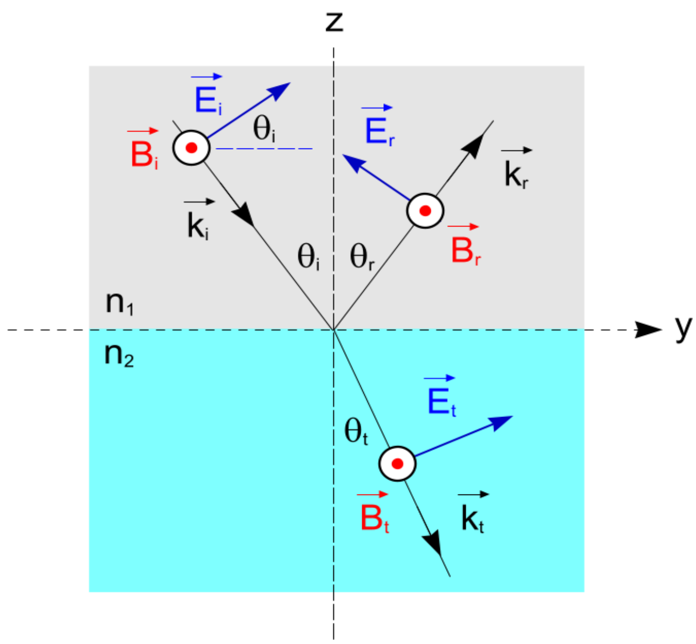
Chopin Third Scherzo 1839 Last Bars	

We now proceed with the beauty of Fresnel's treatment of light incident on an interface. We include the three vectors for each wave: the electric field vector $\vec{E}$, the magnetic field vector $\vec{B}$, and the wave-number vector $\vec{k}$. A "dot" for the magnetic field vector means that the magnetic field points out of the page. An "x" for the magnetic field vector means that the magnetic field points into the page. The electric field and magnetic field are always transverse to the direction of propagation and $\vec{E} \times \vec{B} \sim \vec{k}$. Remember our solution to the wave equation with this condition? Check now that for all three vectors in each case indeed

$$\vec{E} \times \vec{B} \sim \vec{k} .$$

Note: For normal incidence when you go from air to glass there is a hard reflection and the electric field vector flips. The situation is similar to a rope crest flipping to a reflected trough when it meets up with a denser medium. So I chose the reflected electric field to flip.

(p) Polarization



For the tangential components of the electric field to match at the boundary,

$$E_{i0} \cos\theta_i - E_{r0} \cos\theta_r = E_{t0} \cos\theta_t .$$

Next, we consider the tangential components of the magnetic field. The magnetic field vectors **B** are parallel to the interface and perpendicular to the plane of incidence, i.e., the plane of the page. Since they are parallel to the interface,

$$B_{i0} + B_{r0} = B_{t0} .$$

But $\theta_i = \theta_r$ from the law of reflection. So let's call these angles θ_1, angles in the first medium. Then let's call θ_t, the angle in the second medium, θ_2.

Our equations become

$$E_{i0} \cos\theta_1 - E_{r0} \cos\theta_1 = E_{t0} \cos\theta_2 \quad \text{and} \quad B_{i0} + B_{r0} = B_{t0}\text{, i.e.,}$$

$$\cos\theta_1 (E_{i0} - E_{r0}) = E_{t0} \cos\theta_2 \quad \text{and} \quad B_{i0} + B_{r0} = B_{t0}\,.$$

Now we recall the connecting equation $\frac{E_o}{B_o} = c$ in vacuum and $\frac{E_o}{B_o} = v$ in general with $n = \frac{c}{v}$. Therefore, $\frac{B_o}{E_o} = \frac{1}{v} = \frac{n}{c}$ and $B_o = \frac{n}{c} E_o$.

Then $B_{i0} + B_{r0} = B_{t0}$ becomes $\frac{n_1}{c} E_{io} + \frac{n_1}{c} E_{r0} = \frac{n_2}{c} E_{t0}$, or simply

$$n_1 E_{io} + n_1 E_{r0} = n_2 E_{t0} \quad \Rightarrow \quad n_1 (E_{io} + E_{r0}) = n_2 E_{t0}\,.$$

Summary: $\cos\theta_1 (E_{i0} - E_{r0}) = E_{t0} \cos\theta_2$ and $n_1 (E_{io} + E_{r0}) = n_2 E_{t0}$.

We are interested in the reflected amplitude E_{r0} and refracted E_{t0} in terms of E_{i0}.

So we divide our equations by E_{i0} to get

$$\cos\theta_1 \left(1 - \frac{E_{r0}}{E_{i0}}\right) = \frac{E_{t0}}{E_{i0}} \cos\theta_t \quad \text{and} \quad n_1 \left(1 + \frac{E_{r0}}{E_{io}}\right) = n_2 \frac{E_{t0}}{E_{io}}\,.$$

Now we have two equations with two unknowns $\frac{E_{r0}}{E_{io}}$ and $\frac{E_{t0}}{E_{io}}$.

Let's call $\frac{E_{r0}}{E_{io}}$ the reflection coefficient $r = \frac{E_{r0}}{E_{io}}$

and $\dfrac{E_{t0}}{E_{io}}$ the transmission coefficient $t = \dfrac{E_{t0}}{E_{io}}$.

Then our equations are

$$\cos\theta_1(1-r) = t\cos\theta_2 \quad \text{and} \quad n_1(1+r) = n_2 t \ .$$

Note how in physics we often use letters like r and t with different meanings.

Use $t = \dfrac{n_1}{n_2}(1+r)$ from the second equation to eliminate t in the first,

$$\cos\theta_1(1-r) = \frac{n_1}{n_2}(1+r)\cos\theta_2$$

$$\cos\theta_1 - r\cos\theta_1 = \frac{n_1}{n_2}\cos\theta_2 + \frac{n_1}{n_2}r\cos\theta_2$$

$$\cos\theta_1 - r\cos\theta_1 - \frac{n_1}{n_2}r\cos\theta_2 = \frac{n_1}{n_2}\cos\theta_2$$

$$\cos\theta_1 - \frac{n_1}{n_2}\cos\theta_2 = r\cos\theta_1 + \frac{n_1}{n_2}r\cos\theta_2$$

$$n_2\cos\theta_1 - n_1\cos\theta_2 = n_2 r\cos\theta_1 + n_1 r\cos\theta_2$$

$$\boxed{r = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_1\cos\theta_2 + n_2\cos\theta_1}}$$

Then solve for t, using

$$t = \frac{n_1}{n_2}(1+r) \ .$$

$$t = \frac{n_1}{n_2}\left[1 + \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right]$$

$$t = \frac{n_1}{n_2}\left[\frac{(n_1 \cos\theta_2 + n_2 \cos\theta_1) + (n_2 \cos\theta_1 - n_1 \cos\theta_2)}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right]$$

$$t = \frac{n_1}{n_2}\left[\frac{2n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right]$$

$$\boxed{t = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}}$$

Summary:

$$\boxed{r = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1}}$$

$$\boxed{t = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}}$$

What’s this? Fresnel can tell us how much light is reflected by glass?
I remember someone telling me that glass reflects 4% of the light.

Take normal incidence $\theta_1 = 0°$. Then $\theta_2 = 0°$ also, i.e., a normal reflection.

For air $n_1 = 1$ and for glass $n_2 = 1.5$.

$$r = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = \frac{1.5\cos 0° - 1\cos 0°}{1\cos 0° + 1.5\cos 0°} = \frac{1.5 - 1}{1 + 1.5} = \frac{0.5}{2.5}.$$

We find $r = \frac{1/2}{5/2} = \frac{1}{5}$, but we are not finished.

Remember we have to square the amplitude to get energy and/or intensity.

$$R \equiv r^2 = \left[\frac{1}{5}\right]^2 = \frac{1}{25} \qquad \Rightarrow \qquad \frac{1}{25} \times 100\% = 4\%$$

We got the 4% (normal incidence). Therefore, 96% goes into the glass, i.e., is transmitted.

But $$t^2 = \left[\frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right]^2 = \left[\frac{2}{1+1.5}\right]^2 = \left[\frac{2}{2.5}\right]^2 = \left[\frac{2}{5/2}\right]^2$$

$$t^2 = \left[\frac{4}{5}\right]^2 = \frac{16}{25} \neq \frac{24}{25}$$

The math is teaching us that we need to include a factor of $\frac{24}{16} = \frac{3}{2}$.

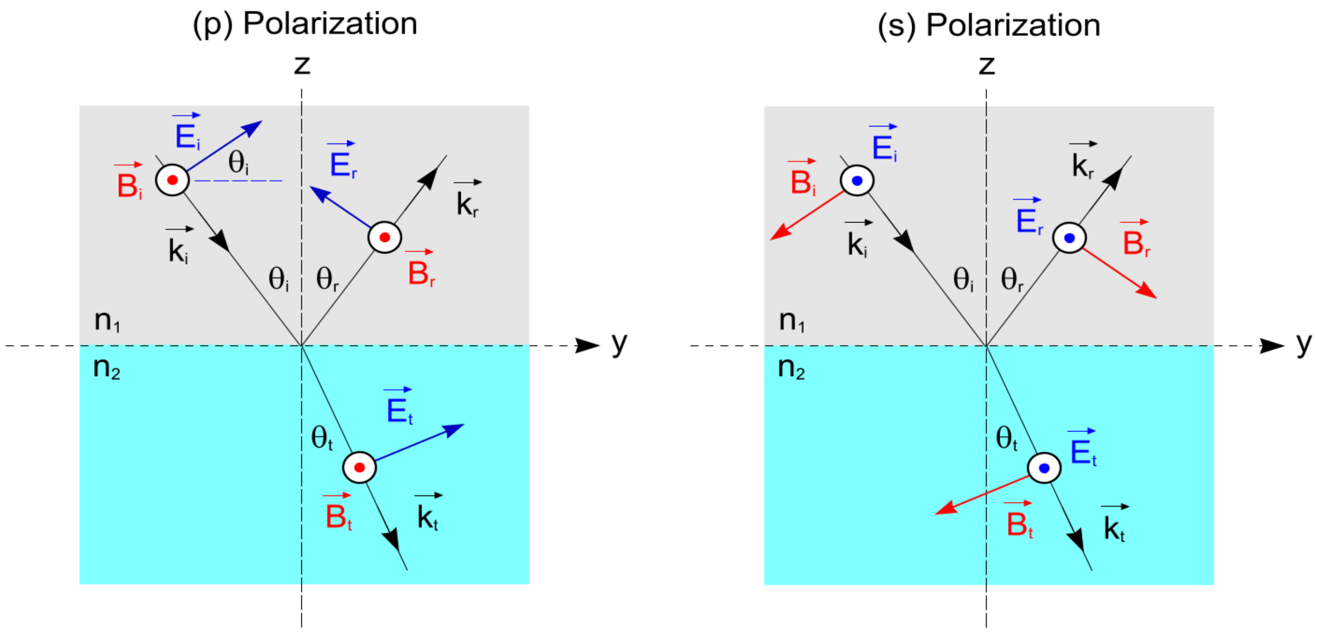
But this factor is n_2, suggesting that the transmitted intensity formula should be

$T = \frac{n_2}{n_1} t^2$ for our analysis of normal incidence.

Since for reflection, you are in the same medium, the factor didn't show up.

The math is telling us that though $I \sim E_o^2$ in a given medium, if you want to compare intensities with different media coming into the picture, you need $I \sim nE_o^2$.

N7. Polarizations (p) and (s). We did the (p) polarization. You will do (s) for homework. The (p) stands for parallel and means that the electric field vector lies in the plane of incidence, defined as the plane in which the incident ray and normal are located. In the figure, this plane is the yz plane. The (s) stands for *senkrecht*, the German for perpendicular. The electric field is perpendicular to the plane of incidence. The word for parallel in German is the same as in English.



We will emphasize now that we did the parallel case by including a subscript p.

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \qquad t_p = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}$$

N8. Fresnel Equations in Terms of θ_1. We can use Snell's law to eliminate θ_2

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

$$\cos\theta_2 = \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}$$

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \quad => \quad r_p = \frac{n_2 \cos\theta_1 - n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}}{n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2} + n_2 \cos\theta_1}$$

$$t_p = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \quad => \quad t_p = \frac{2n_1 \cos\theta_1}{n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2} + n_2 \cos\theta_1}$$

The reflection parameter R is called the reflectivity.

The transmitted parameter T is called the transmissivity.

In terms of the initial angle of incidence and indexes of refraction these quantities are below.

$$R_p = \left[\frac{n_2 \cos\theta_1 - n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}}{n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2} + n_2 \cos\theta_1} \right]^2$$

$$T_p = 1 - R_p$$

The s polarization equations, which you will derive, are given below.

$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \quad \text{and} \quad t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

OVERALL SUMMARY

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \qquad t_p = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}$$

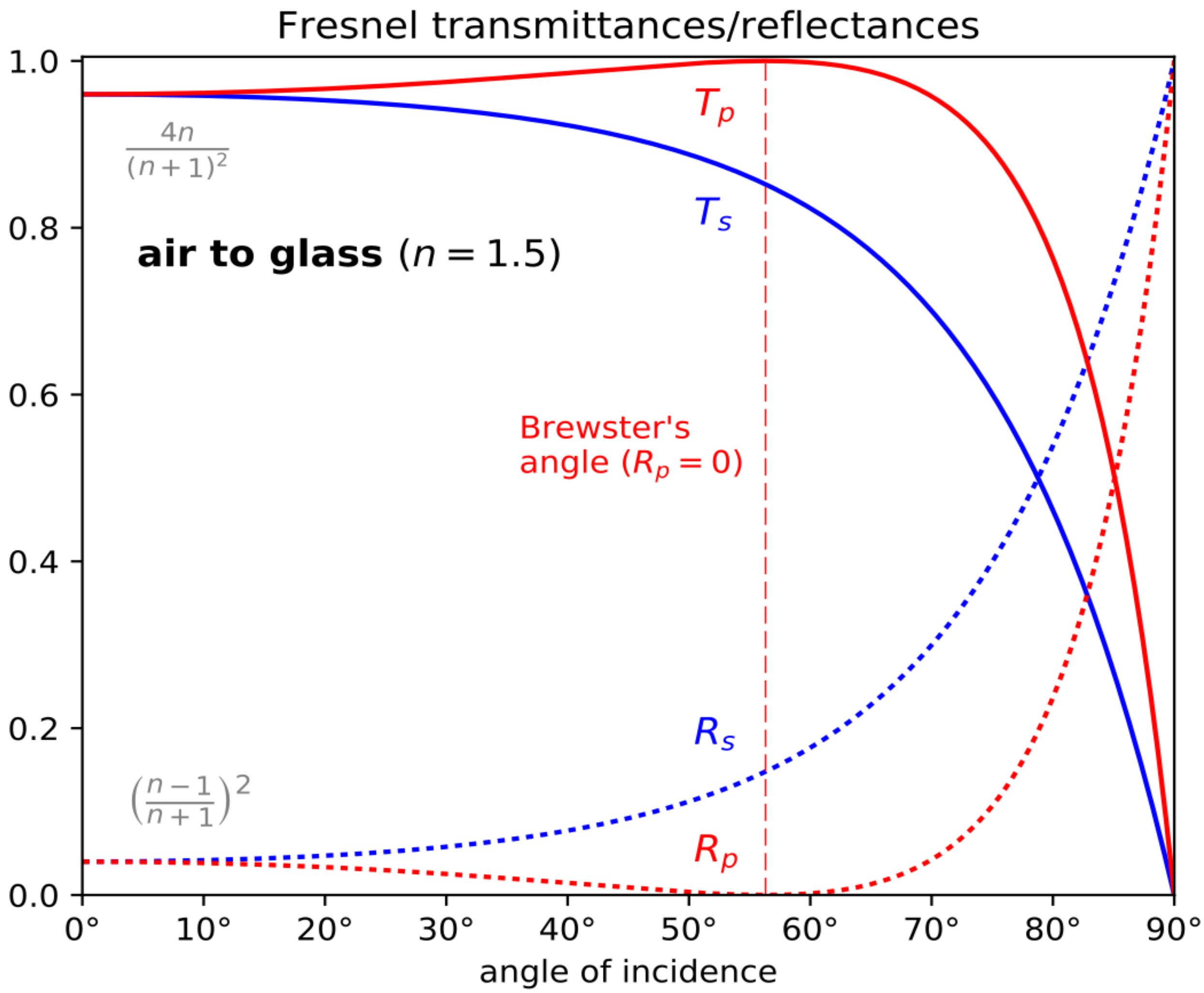
$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \qquad t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$R_p = \left[\frac{n_2 \cos\theta_1 - n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}}{n_1 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2} + n_2 \cos\theta_1} \right]^2 \qquad T_p = 1 - R_p$$

$$R_s = \left[\frac{n_1 \cos\theta_1 - n_2 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}}{n_1 \cos\theta_1 + n_2 \sqrt{1 - (\frac{n_1}{n_2} \sin\theta_1)^2}} \right]^2 \qquad T_s = 1 - R_s$$

Since natural light is unpolarized, we have a mixture of the p and s polarizations.

$$R = \frac{R_p + R_s}{2} \qquad T = 1 - R$$



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$$R_p = \left[\frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_1\cos\theta_2 + n_2\cos\theta_1}\right]^2 \xrightarrow[\text{glass}]{\text{air to}} \left[\frac{n\cos\theta_1 - \cos\theta_2}{\cos\theta_2 + n\cos\theta_1}\right]^2 \xrightarrow[\text{incidence}]{\text{normal}} \left[\frac{n-1}{n+1}\right]^2$$

$$R_s = \left[\frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}\right]^2 \xrightarrow[\text{glass}]{\text{air to}} \left[\frac{\cos\theta_1 - n\cos\theta_2}{\cos\theta_1 + n\cos\theta_2}\right]^2 \xrightarrow[\text{incidence}]{\text{normal}} \left[\frac{n-1}{n+1}\right]^2$$

$$T_{\text{p,s}} \xrightarrow[\text{incidence}]{\text{normal}} = 1 - \left[\frac{n-1}{n+1}\right]^2 = \frac{(n+1)^2 - (n-1)^2}{(n+1)^2} = \frac{4n}{(n+1)^2}$$

Brewster’s angle is to be discussed in the next chapter.