Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), doctorphys on YouTube
Chapter H. Conservation of Energy. Prerequisite: Calculus I. Corequisite: Calculus II.

H1. Conservative Forces. One definition we can use for a *conservative force* is

> Definition 1. Conservative Force – a force that acts on a body in such that when the body returns to its original position, it has the same kinetic energy it started with.

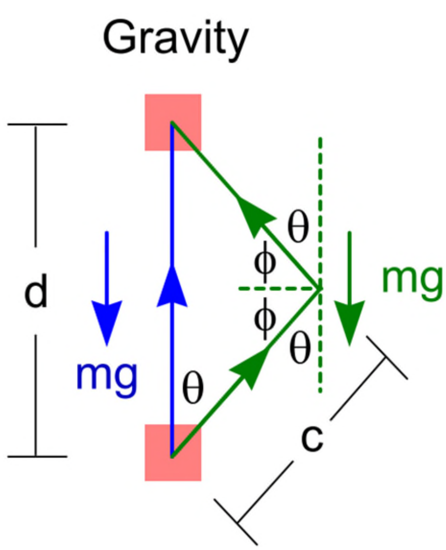
Two examples are:

> 1. Gravity – "What goes up, must come down."
> 2. Spring – "What goes in, must come out."

Friction would mess these situations up. So we are considering ideal cases where there is no friction. If a ball is tossed straight up in the air and there is no friction, the ball will return to its original position with the speed it started with, though now heading downward instead of upward. But since the kinetic energy, $K = \frac{1}{2}mv^2$, involves squaring the speed, the kinetic energy is the same at the beginning and end of the trip.

A similar situation occurs with a spring obeying Hooke's Law with no friction. If you smack a mass on a spring so that it has an initial speed to compress the spring, it will return to its initial position with the same speed moving outward. In either case, you get back the kinetic energy that you started with. Note that with frictional forces we DO NOT get back what we started with. Therefore, frictional forces are not conservative. Here is another definition.

> Definition 2. Conservative Force – a force such that the work needed to move an object from point 1 to point 2 is independent of the path taken.



Let's check. First take the path straight up, the blue path. We need to gently lift the block with a force to counteract downward gravity. So our force is upward and equal to mg. The work done to move the ball up a distance d is then

$$W_{\text{blue}} = Fd = mg(\cos 0°)d = mgd \ .$$

For the green path we still lift gently with force mg, but now we are at an angle.

$$W_{\text{green}} = mg(\cos\theta)c + mg[\cos(2\phi+\theta)]c$$

The angles $2\phi+\theta$ and θ are supplementary.

Therefore, they have the same cosine.

$$\cos(\theta + 2\phi) = \cos(\theta) \ .$$

The work along the green path is then

$$W_{\text{green}} = 2mg(\cos\theta)c \ .$$

But $(\cos\theta)c = \dfrac{d}{2}$. Therefore,

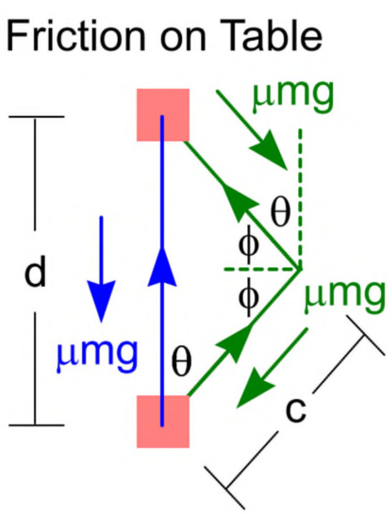
$$W_{\text{green}} = 2mg(\cos\theta)c = 2mg\frac{d}{2} = mgd \ ,$$

and we have

$$W_{\text{blue}} = W_{\text{green}} \ .$$

The work is independent of path.

Now we try a similar calculation with moving a block on a table where there is friction.



We must counteract friction as we move the block. For the blue path

$$W_{\text{blue}} = Fd = \mu mgd \ .$$

For the green path, note that the friction points opposite to our path again. Therefore, we need to apply a force of μmg along the distance c two times.

$$W_{\text{green}} = 2\mu mg(\cos 0°)c = 2\mu mgc$$

Since $d \neq 2c$,

$$W_{\text{blue}} \neq W_{\text{green}} \ .$$

The work is NOT path independent.

Therefore, frictional forces are not conservative.

H2. Potential Energy.

1. Gravity. We first consider the conservative force of gravity. If we do work to gently lift up a ball a height h, we do work in the amount of

$$W = mgh.$$

As we hold the ball up there, we know that if we release it, gravity will do the same work in bringing the ball down. The ball will speed up as it falls and we will find

$$W = mgh = \frac{1}{2}mv^2$$

by the Work-Energy Theorem.

When we are holding the ball up there before release, we can say we have *potential energy* and when we release the ball, that potential energy is transformed into *kinetic energy*. Before we drop the ball, the energy is all potential. And after the ball falls, just before it hits the ground, the energy is all kinetic.

$$E_{\text{before}} = mgh$$

$$E_{\text{after}} = \frac{1}{2}mv^2$$

The energy before and after are equal. We say energy is conserved. We have a new principle, the *conservation of energy*, which is the title of our chapter.

$$E_{\text{before}} = E_{\text{after}}$$

When the ball is on its way down, it has a combination of some potential energy and some kinetic. Therefore, in general, we can write the total energy as a sum.

$$\boxed{E = \frac{1}{2}mv^2 + mgh}$$

Let the kinetic energy be represented by $K = \frac{1}{2}mv^2$ and the potential energy be designated as $U = mgh$. Then, the total energy is given by as follows.

$$\boxed{E = K + U}$$

$$\boxed{U = mgh}$$

Since the potential energy is intimately related to the conservative force, we would like a relationship between the force and the potential energy. Choosing up as the positive direction with coordinate z and the ground being z = 0. Then, the force due to gravity is

$$F = -mg \ .$$

The potential energy we defined above is

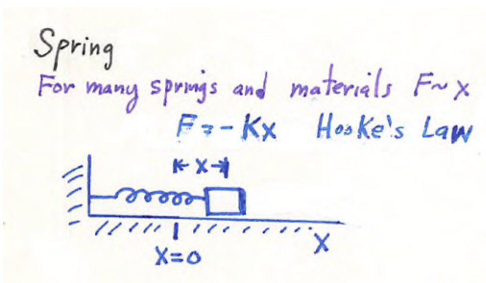
$$U = mgz \ .$$

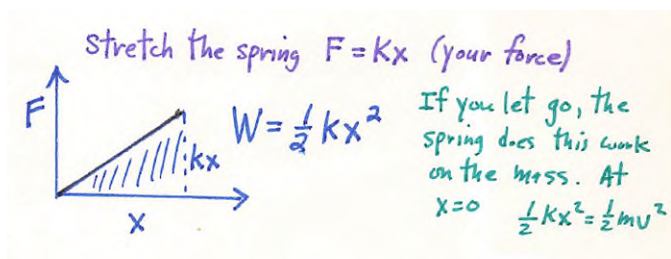
How are they related?

The answer is negative the derivative.

$$\boxed{F = -\frac{dU}{dz}}$$

2. Spring. Next is the spring that obeys Hooke's Law.





With gravity, we picked up a ball and let it go. With the spring we pull the block in the figure and let it go. We saw in the previous chapter that the work is given by the area under the Hooke's Law graph.

$$W = \frac{1}{2}kx^2$$

By analogy with gravity, this work is the potential energy for the spring.

$$\boxed{U = \frac{1}{2}kx^2}$$

Do we get the spring force if we take negative the derivative? Yes!

$$F = -\frac{dU}{dx} = -\frac{d}{dx}(\frac{1}{2}kx^2) = -\frac{1}{2}k\frac{d}{dx}x^2 = -\frac{1}{2}k(2x) = -kx$$

H3. Conservation of Energy.

1. Conservative Forces. For conservative forces we can write the conservation of energy as

$$E = K + U = \text{const}\,,$$

where const stands for constant.

Therefore changes in total energy E are zero.

$$\Delta E = \Delta K + \Delta U = 0$$

The changes in kinetic energy and potential energy are opposite each other:

$$\Delta K = -\Delta U \quad \text{and} \quad \Delta U = -\Delta K\,.$$

Since the energy is constant we can write for two different locations

$$E_1 = E_2 \quad \text{or} \quad E_{before} = E_{after}\,, \quad \text{and}$$

$$K_1 + U_1 = K_2 + U_2\,,$$

$$K_{before} + U_{before} = K_{after} + U_{after}\,.$$

These equations prove very helpful in solving problems, as you will see.

2. Nonconservative Forces. Suppose we have a problem with friction. Friction always works against you, so the work is negative.

$$W_f = -fd < 0$$

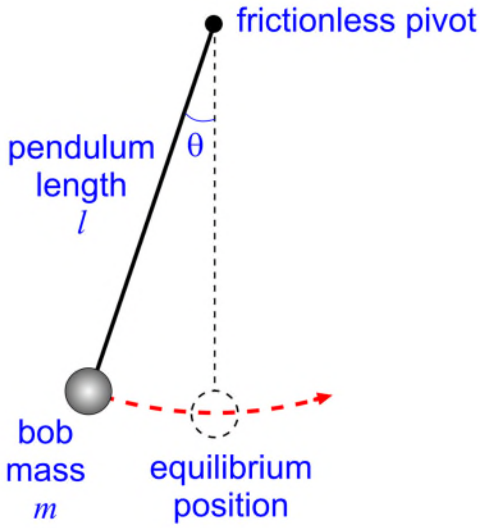
We need to put this component in by hand. Here is how I do it.

$$K_1 + U_1 - fd = K_2 + U_2$$

Some of the energy goes into heat as the surfaces rub. This transfer of energy into heat means that the available energy for kinetic and potential at location 2 is reduced. But this frictional work goes into heat. When heat is considered, the total energy is still conserved!

One final comment. There used to be two conservations laws: one for energy and one for matter. Since Einstein showed us that energy and matter are equivalent in the form $E = mc^2$, we can generalize our conservation of energy to the ***conservation of matter-energy***.

H4. Pendulum.



Adaptation of a Figure
Courtesy Chetvorno, Wikimedia
Released into the Public Domain

Pull back the bob (mass $m = 2\text{ kg}$) back on a pendulum of length $l = 1.5\text{ m}$ so that the angle is $\theta = 90°$. Release the bob from rest.

(i) What is the speed when the bob reaches the bottom of the pendulum?

(ii) Derive a general formula for the speed at the bottom if the bob is released from rest at θ.

(i) We do not need to draw a force diagram in this case. The conservation of energy will do the trick. We are always free to pick our reference. I will pick the zero reference height to be at the bottom of the pendulum, the equilibrium position.

The initial height of release is then

$$h = l\ .$$

The equations for the conservation of energy can be written as

$$K_{before} + U_{before} = K_{after} + U_{after}$$

$$0 + mgl = \frac{1}{2}mv^2 + 0$$

Solve for the speed v.

$$\frac{1}{2}mv^2 = mgl$$

$$\frac{1}{2}v^2 = gl \quad => \quad v^2 = 2gl$$

$$v = \sqrt{2gl}$$

The answer is independent of the mass. It depends on the pendulum length and planet.

$$v = \sqrt{2(9.8\frac{\text{m}}{\text{s}^2})(1.5\text{ m})} = \sqrt{29.4\frac{\text{m}^2}{\text{s}^2}} = 5.4\frac{\text{m}}{\text{s}}$$

(ii) General formula for the speed at the bottom if the bob is released from rest at θ .

The initial height of release is now

$$h = l - l\cos\theta = l(1-\cos\theta) .$$

The equations for the conservation of energy can be written as

$$K_{before} + U_{before} = K_{after} + U_{after}$$

$$0 + mgl(1-\cos\theta) = \frac{1}{2}mv^2 + 0$$

Solve for the speed v .

$$\frac{1}{2}mv^2 = mgl(1-\cos\theta)$$

$$\frac{1}{2}mv^2 = mgl(1-\cos\theta)$$

$$v^2 = 2gl(1-\cos\theta)$$

$$\boxed{v = \sqrt{2gl(1-\cos\theta)}}$$

Checks. If you release from $\theta = 0°$, you get

$$v = \sqrt{2gl(1-\cos\theta)} = \sqrt{2gl(1-\cos 0°)} = \sqrt{2gl(1-1)} = 0$$

If you release from $\theta = 90°$, you get

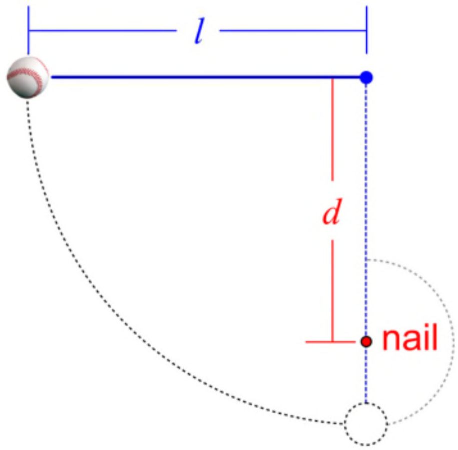
$$v = \sqrt{2gl(1-\cos\theta)} = \sqrt{2gl(1-\cos 90°)} = \sqrt{2gl(1-0)} = \sqrt{2gl} \ .$$

But this result is the same for falling straight down a distance $h = l$:

$$mgl = \frac{1}{2}mv^2 \quad => \quad gl = \frac{1}{2}v^2 \quad => \quad 2gl = v^2 \quad => \quad v = \sqrt{2gl} \ .$$

The falling in an arc does not change the result since the pendulum rope or cable does not do any work on the bob. The reason the rope does not work is that the tension in the rope acts perpendicular to the direction of the speed. Therefore, the work for each small movement Δs along the arclength is $\Delta W_{rope} = T(\cos 90°)\Delta s = 0$. The rope serves only to change the direction.

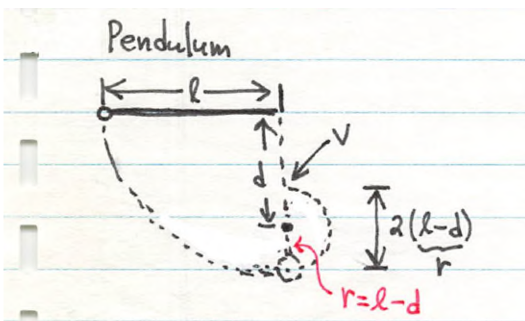
H5. Pendulum and Nail.



Baseball Courtesy Tage Olsin CC-BY-SA-2.0

I encountered this problem in a 1978 edition of the legendary Halliday and Resnick text during my first year of teaching that same year. It was their third edition. I had studied in college with their 2nd edition.

A bob on a pendulum of length l swings down from 90°. The rope of the pendulum encounters a nail when the bob reaches its lowest point. The lower portion of the rope then swings in a smaller circle. (i) Where should the nail be placed from the ceiling so that the bob released from rest at the ceiling just reaches the apex on the smaller circle. If the nail is any higher, the bob will not reach the apex. Give you answer as a fraction of the pendulum length l. (ii) Find the simplest formula for the velocity v at the apex of the smaller circle.



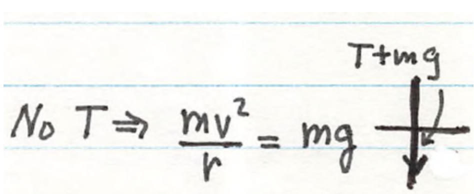
(i) What is d? We introduce the radius r of the smaller circle and relate it to the pendulum length l and the distance d the nail is from the ceiling. We choose the reference to be zero at the lowest point of the pendulum. Conservation of energy gives

$$K_{before} + U_{before} = K_{after} + U_{after} ,$$

where "before" will represent the ball at the top about to be released from the ceiling and "after" is the location at the apex of the smaller circle.

$$0 + mgl = \frac{1}{2}mv^2 + mg[2(l-d)]$$

But we immediately encounter a difficulty. How do we get rid of the velocity v if we are to solve for d. Here is where we use the information that the bob just reaches the apex of the smaller circle. Now we could use a force diagram.



At the apex, the two forces on the bob are the tension in the rope and gravity.

$$\sum F = T + mg = ma = m\frac{v^2}{r}$$

For the bob to just make it to the apex, the tension will be zero: $T = 0$, giving $mg = m\frac{v^2}{r}$. From the figure you can see that $r = l - d$. Therefore, $mg = mv^2 / (l - d)$.

Our two main equations are now

$$mgl = \frac{1}{2}mv^2 + mg[2(l-d)],$$

$$mg = \frac{mv^2}{l-d}.$$

Rearranging things slightly, we obtain

$$mgl = \frac{1}{2}mv^2 + 2mg(l-d),$$
$$\frac{1}{2}mv^2 = \frac{1}{2}mg(l-d).$$

Substitute $\frac{1}{2}mv^2 = \frac{1}{2}mg(l-d)$ into the first equation.

$$mgl = \frac{1}{2}mg(l-d) + 2mg(l-d),$$

$$mgl = \frac{5}{2}mg(l-d)$$

Solve for d in terms of l as we were instructed.

The masses cancel and so does gravity.

$$l = \frac{5}{2}(l-d)$$

$$\frac{2}{5}l = l-d$$

$$d = l - \frac{2}{5}l$$

$$\boxed{d = \frac{3}{5}l}$$

Also acceptable is the answer $d = 0.6l$.

(ii) Find the simplest formula for the velocity v at the apex of the smaller circle. We have two equations the contain the velocity v.

$$mgl = \frac{1}{2}mv^2 + 2mg(l-d) \text{ ,}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mg(l-d) \text{ .}$$

The second equation looks simpler.

$$\frac{1}{2}mv^2 = \frac{1}{2}mg(l-d)$$

$$v^2 = g(l-d)$$

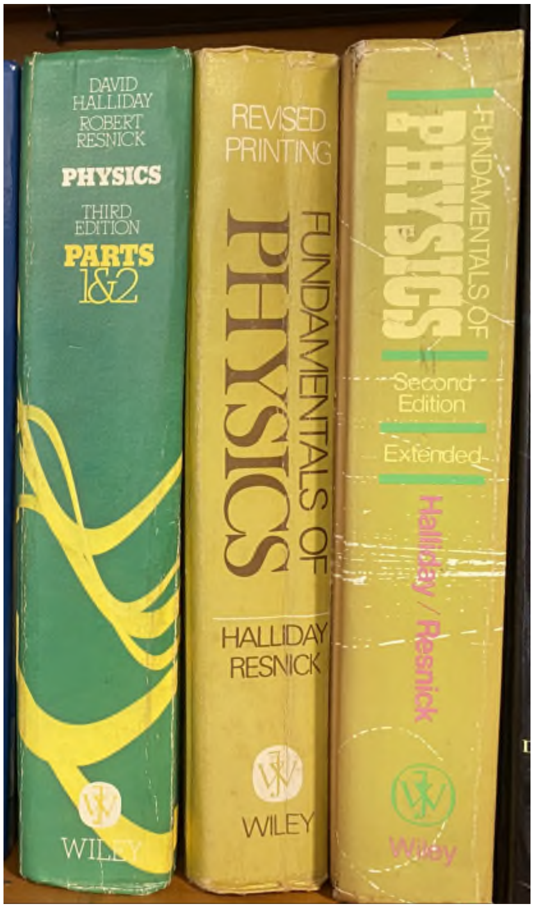
We know from part (i) that $d = \frac{3}{5}l$.

$$v^2 = g(l - \frac{3}{5}l)$$

$$v^2 = g(\frac{2}{5}l)$$

$$v^2 = \frac{2}{5}gl$$

$$\boxed{v = \sqrt{\frac{2}{5}gl}}$$



H6. Sliding Off Ice Hemisphere. Here is another problem from my first year of teaching, where I used the 3rd edition of Halliday and Resnick. It has the green cover in the photo shown at the left.

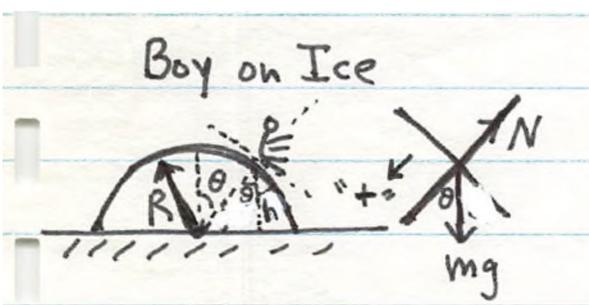
A boy is sitting on an ice-mound hemisphere (or a hemisphere of packed snow) with radius R. At first, the boy is sitting at the top of the mound of ice-snow; then the boy starts to slide down, with friction taken to be zero. At some point the boy leaves the mound of snow.

The original problem only asked for (i) below, but I am adding two additional questions. No numbers were given in the original problem.

(i) Find the height h from the ground where the boy loses contact with the snow.

(ii) Find the angle with respect to the vertical where the boy leaves the mound of snow.

(iii) Find the speed at which the child leaves the snow.



(i) Find h. The key here is that we need a force diagram because when the boy leaves the ice, the normal force N is zero. There is no more contact between the boy and the ice.

This technique of conservation of energy and a force diagram is one we saw in the previous section. It is nice how physics builds on prior knowledge and methods. In this problem we again get to review our force diagram type of problem with Newton's Second Law with circular motion, as well as use conservation of energy.

We will start first with the force diagram:

$$mg\cos\theta - N = m\frac{v^2}{R},$$

where $N = 0$.

The result is $mg\cos\theta = m\dfrac{v^2}{R}$.

For conservation of energy, pick the point at the top as location 1 and where the boy flies off the mound as location 2. Refer to the above figure.

$$0 + mgR = \frac{1}{2}mv^2 + mgh$$

From the geometry, we can substitute $h = R\cos\theta$ in the above equation. The result is

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$

Now, here is a trick I like to use in situations like these. We have two main equations.

$$mg\cos\theta = m\frac{v^2}{R}$$

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$

I like to get both equations with mv^2 by multiplying accordingly for each equation.

$$mgR\cos\theta = mv^2$$

$$2mgR = mv^2 + 2mgR\cos\theta$$

Next I substitute the $mv^2 = mgR\cos\theta$ from the first equation into the second.

$$2mgR = mgR\cos\theta + 2mgR\cos\theta$$

At this point, we can divide out the mgR combination.

$$2 = \cos\theta + 2\cos\theta$$

It looks like we are going to solve for the angle first.

$$2 = 3\cos\theta$$

$$\cos\theta = \frac{2}{3}$$

$$\theta = 48.19°$$

We know that $h = R\cos\theta$, so the height follows readily.

$$h = R\cos\theta$$

Since $\cos\theta = \frac{2}{3}$, we do not even need the actual angle to find h.

$$h = R\cos\theta = R\cdot\frac{2}{3}$$

$$\boxed{h = \frac{2}{3}R}$$

(ii) The Angle θ. We already found the angle in our work for Part (i).

$$\boxed{\theta = 48.19^\circ}$$

(iii) The Velocity v. We can look back and pick out a formula with v. Here is one below.

$$mgR\cos\theta = mv^2$$

Since $\cos\theta = \frac{2}{3}$ and the mass divides out, we quickly obtain

$$gR\frac{2}{3} = v^2$$

$$\boxed{v = \sqrt{\frac{2gR}{3}}}$$

Just to get an idea with some numbers, take a radius $R = 1.25\ \mathrm{m}$. Then

$$v = \sqrt{\frac{2gR}{3}} = \sqrt{\frac{2\cdot(9.8)\cdot(1.25)}{3}} = \sqrt{8.167} = 2.86\ \frac{\mathrm{m}}{\mathrm{s}}$$

This speed is also equal to $10.3\ \frac{\mathrm{km}}{\mathrm{h}} = 6.40\frac{\mathrm{mi}}{\mathrm{h}}$.

H7. Loop-the-Loop. The classic loop-the-loop problem is very popular and an excellent application to a real-life problem. Below is the “Olympia Looping” ride in .

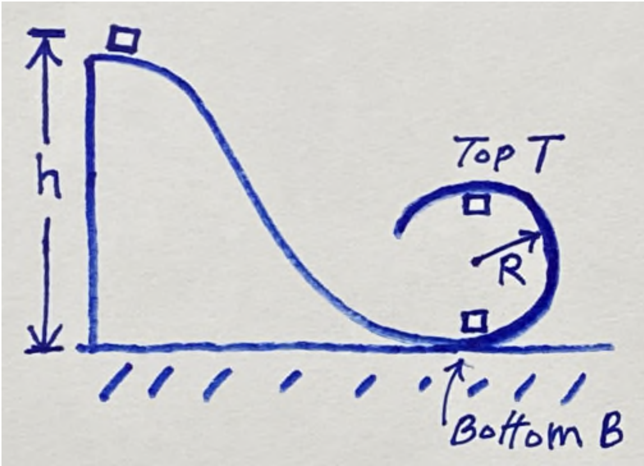
Olympia Looping, Photo Courtesy Oliver Mallich, flickr, [License Attribution-NoDerivs 2.0]
Oktoberfest, Munich, Bavaria, Germany, October 2, 2006

The “Olympia Looping” is a portable roller coaster! “It is the largest portable roller coaster in the world, and the only one with five inversions. It appears at many carnivals in Germany, most notably Oktoberfest, where it made its debut in 1989.” Wikipedia

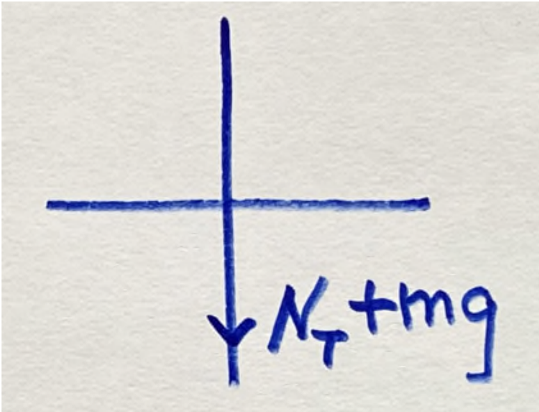
The roller coaster is also known as “Munich Looping.” Notice the inverted teardrop shape rather than a completely circular path. This deviation from an exact circle is important as we will show that a purely circular structure has dangerous g-forces at the bottom. The bottom of the loops in the above photo are not horizontal at the bottom. This feature allows for reduced g-forces at the bottom where the coaster is zipping along pretty fast.

We are going to analyze the perfect circular loop with no friction in this section.

Problem: Loop-the-Loop. A roller coaster rolls down from a height h and enters a loop-the-loop at ground level. The loop is a circle with radius R. (i) What must the initial height h be for the coaster so that when it reaches the top of the loop-the-loop there is no normal force on the coaster. The coaster can be taken to have tiny wheels. Later in our course, I will show you that a car with tiny wheels rolls like a sliding frictionless mass. So, for all practical purposes, you are doing the equivalent problem for a sliding block with no friction. (ii) Give the g-force at the bottom of the loop.



Solution: Loop-the-Loop. (i) Find h. We start with the top location of the loop.



$$N_T + mg = m\frac{v_T^2}{R}$$

The normal force is zero at this point: $N_T = 0$, giving

$$mg = m\frac{v_T^2}{R}$$

Conservation of energy from the very beginning to this point gives

$$0 + mgh = \frac{1}{2}mv_T^2 + mg(2R),$$

where the kinetic energy at the very start is zero with height h.

I like to get rid of the denominators in each equation.

$$mg = m\frac{v_T^2}{R} \quad => \quad mgR = mv_T^2$$

$$mgh = \frac{1}{2}mv_T^2 + mg(2R) \quad => \quad 2mgh = mv_T^2 + 2mg(2R)$$

Next comes substituting $mv_T^2 = mgR$ into $2mgh = mv_T^2 + 2mg(2R)$.

$$2mgh = mgR + 2mg(2R)$$

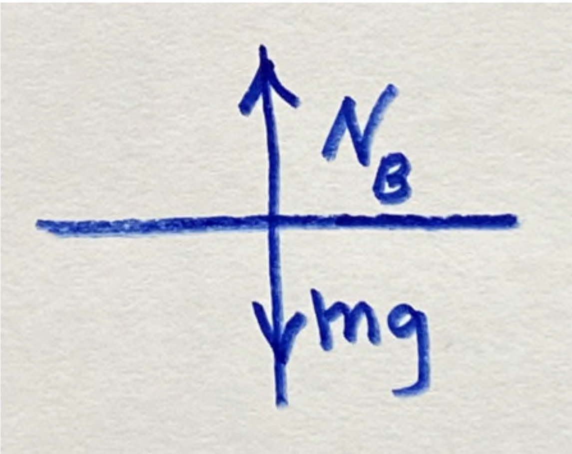
The mg divides out.

$$2h = R + 2(2R) \quad => \quad 2h = R + 4R$$

$$2h = 5R$$

$$\boxed{h = \frac{5}{2}R}$$

(ii) Find N_B at the bottom. The normal gives you what a scale underneath you at the bottom would read. For the bottom location, the force diagram leads to



$$N_B - mg = m\frac{v_B^2}{R}.$$

Conservation of energy gives

$$0 + mgh = \frac{1}{2}mv_B^2 + 0.$$

But we know from (i) that $h = \frac{5}{2}R$,

Our two equations with this substitution in the second equation are then

$$N_B - mg = m\frac{v_B^2}{R}$$

$$mg\frac{5}{2}R = \frac{1}{2}mv_B^2$$

We want to solve for N_B. From the first equation, $N_B = m\frac{v_B^2}{R} + mg$.

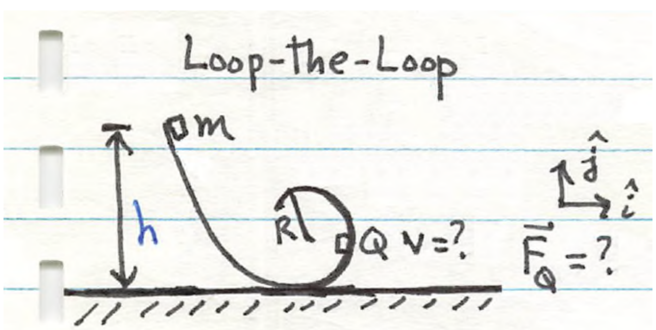
The second equation $mg\frac{5}{2}R = \frac{1}{2}mv_B^2$ can be put in the form $5mg = m\frac{v_B^2}{R}$. Then

$$N_B = m\frac{v_B^2}{R} + mg = 5mg + mg.$$

$\boxed{N_B = 6mg}$ This is $6g$!!!

Therefore, they do not construct a perfectly circular shape at the bottom. See below. The loops have inverted tear-drop shapes.

Olympia Looping, Photo Courtesy Bjs, Wikimedia, Dedicated to the Public Domain
Oktoberfest, Munich, Bavaria, Germany, September 2005

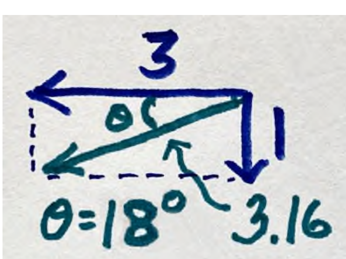


Here is an additional problem for you. Find the force at point Q, using our previous result that $h = \frac{5}{2}R$. See if you get the answer

$$\vec{F}_Q = -3mg\hat{i} - mg\hat{j},$$

a g-force of 3g to the center and 1g down.

The magnitude of the total is $a_Q = \sqrt{3^2 + 1^2} = \sqrt{10} = 3.16g$. What is the direction?

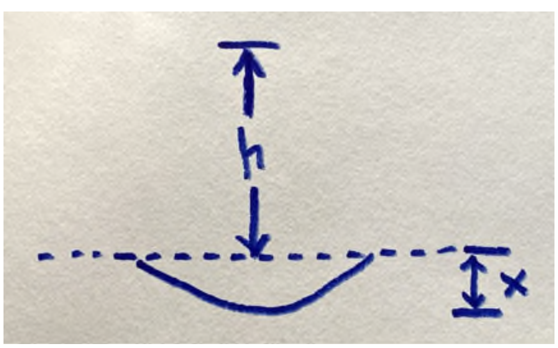


The answer is $\theta = \tan^{-1}\frac{1}{3} = 18°$, South of due West.

H8. The Trampoline.

Courtesy henry…, flickr. License: Attribution-NonCommercial-NoDerivs 2.0

Problem. A person with mass m is jumping on a trampoline, landing on its center each time. Assume that the trampoline obeys Hooke's Law with a spring constant k in its center. What is the maximum depression x of the trampoline measured downward from the trampoline's flat equilibrium surface if the person falls from a maximum height h above the trampoline on every bounce?



Solution.

$$K_{top} + U_{top} = K_{bottom} + U_{bottom}$$

$$0 + mg(h + x) = 0 + \frac{1}{2}kx^2$$

Solve for x.

$$mg(h + x) = \frac{1}{2}kx^2$$

$$mgh + mgx = \frac{1}{2}kx^2 \quad => \quad 2mgh + 2mgx = kx^2 \quad => \quad kx^2 - 2mgx - 2mgh = 0$$

We have a quadratic equation.

$$kx^2 - 2mgx - 2mgh = 0$$

A quadratic equation of the form

$$ax^2 + bx + c = 0$$

has solutions

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For our equation $a = k$, $b = -2mg$, and $c = -2mgh$.

The two solutions in our case are then

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad => \quad \frac{-(-2mg) \pm \sqrt{(-2mg)^2 - 4(k)(-2mgh)}}{2k}$$

$$=> \quad \frac{2mg \pm \sqrt{(2mg)^2 + 4(k)(2mgh)}}{2k}.$$

Since $x > 0$, we want

$$x = \frac{2mg + \sqrt{(2mg)^2 + 4(k)(2mgh)}}{2k} \quad => \quad x = \frac{mg + \sqrt{(mg)^2 + (k)(2mgh)}}{k}$$

$$=> \quad x = \frac{mg + mg\sqrt{1 + \frac{2kh}{mg}}}{k}$$

$$\boxed{x = \frac{mg}{k}\left[1 + \sqrt{1 + \frac{2kh}{mg}}\right]}$$

Is the answer reasonable?

The units check out, but when $h = 0$, why is $x = \frac{2mg}{k}$ and not $x = \frac{mg}{k}$?

I will address this issue in the accompanying video lecture at
https://www.youtube.com/doctorphys.