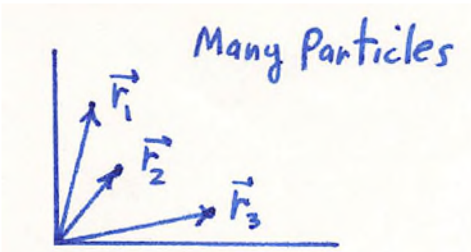
I1. The Center of Mass. The concept of the center of mass will arise naturally as we apply Newton's Second Law to a system of particles or bodies. The figure below shows three particles and their positions described by three vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.



Imagine more particles so that there are n of them. We can write Newton's Second Law for each particle, where each force represents the net force on each particle.

$$\vec{F}_1 = m_1\vec{a}_1 \qquad \vec{F}_2 = m_2\vec{a}_2 \qquad \vec{F}_3 = m_3\vec{a}_3 \qquad \ldots \qquad \vec{F}_n = m_n\vec{a}_n$$

Now add these equations.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n$$

All of the internal forces that the masses exert on each other will cancel according to Newton's Third Law of Action-Reaction. We will be left with external forces acting on the masses. Let the sum of the external forces be $\vec{F}_{ext}$. Our equation then becomes

$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n \ .$$

We would like $\vec{F}_{ext} = M\vec{a}$, where

$$M = m_1 + m_2 + m_3 + \ldots + m_n$$

is the total mass. What is the special $\vec{a}$ that results?

We will call it the acceleration of the center of mass. Then

$$\vec{F}_{ext} = M\vec{a}_{cm} \text{ with } M = m_1 + m_2 + m_3 + \ldots + m_n \ .$$

At this point we use the general fact that $\vec{a} = \dfrac{d\vec{v}}{dt}$ to obtain

$$\vec{F}_{ext} = M\frac{d\vec{v}_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \ldots + m_n\frac{d\vec{v}_n}{dt} \ ,$$

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n \ .$$

Next we use the general fact that $\vec{v} = \frac{d\vec{r}}{dt}$ to get

$$M \frac{d\overrightarrow{r_{cm}}}{dt} = m_1 \frac{d\vec{r_1}}{dt} + m_2 \frac{d\vec{r_2}}{dt} + m_3 \frac{d\vec{r_3}}{dt} + \ldots + m_n \frac{d\vec{r_n}}{dt} ,$$

$$M \overrightarrow{r_{cm}} = m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3} + \ldots + m_n \vec{r_n} .$$

The center of mass is defined as $\overrightarrow{r_{cm}}$.

$$\boxed{\overrightarrow{r_{cm}} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3} + \ldots + m_n \vec{r_n}}{m_1 + m_2 + m_3 + \ldots + m_n}}$$

We can express the vector $\overrightarrow{r_{cm}}$ in terms of its components

$$\overrightarrow{r_{cm}} = x_{cm} \hat{i} + y_{cm} \hat{j} + z_{cm} \hat{k} ,$$

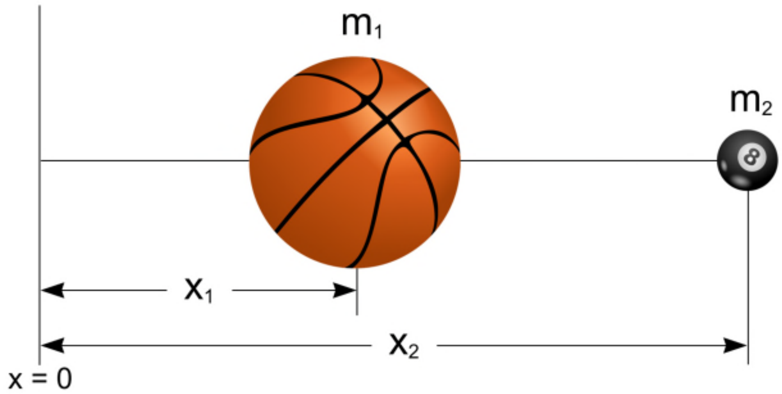
where x_{cm} , y_{cm} , and z_{cm} are given below.

$$\boxed{x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n}}$$

$$\boxed{y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots + m_n y_n}{m_1 + m_2 + m_3 + \ldots + m_n}}$$

$$\boxed{z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots + m_n z_n}{m_1 + m_2 + m_3 + \ldots + m_n}}$$

I2. The Center of Mass of Two Masses. Find the center of mass for the pair below.



Images Courtesy publicdomainvectors.org

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

What happens if $m_1 = m_2 = m$?

$$x_{cm} = \frac{m x_1 + m x_2}{m + m} = \frac{x_1 + x_2}{2}$$

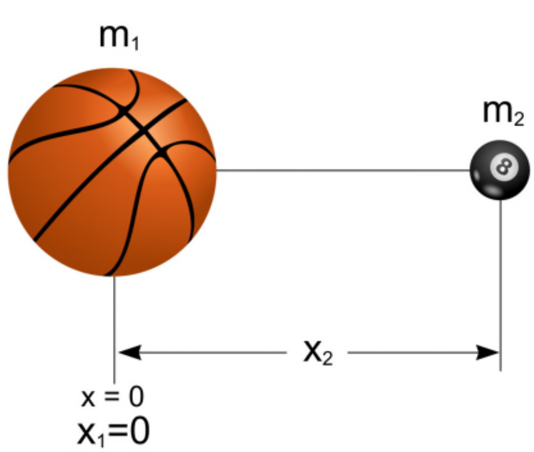
The center of mass is exactly between the two masses. Think of it as a balance point. What about the result for a real basketball and pool ball? The mass of our basketball is about 0.62 kg. We will use $m_1 = 0.64 \text{ kg}$ so the mass is four times the pool ball mass, as a typical pool ball has a mass $m_2 = 0.16 \text{ kg}$. The answer is

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{(0.64 \text{ kg}) x_1 + (0.16 \text{ kg}) x_2}{0.64 \text{ kg} + 0.16 \text{ kg}} = \frac{0.64 x_1 + 0.16 x_2}{0.80} = \frac{4}{5} x_1 + \frac{1}{5} x_2 \,.$$

Where is this point? It is

$$x_{cm} - x_1 = \frac{4}{5} x_1 + \frac{1}{5} x_2 - x_1 = -\frac{1}{5} x_1 + \frac{1}{5} x_2 = \frac{1}{5}(x_2 - x_1)\,,$$

i.e., to the right of the basketball center at 1/5 the distance between the mass centers.



It is convenient to choose the origin at the center position of the first mass.

Images Courtesy publicdomainvectors.org

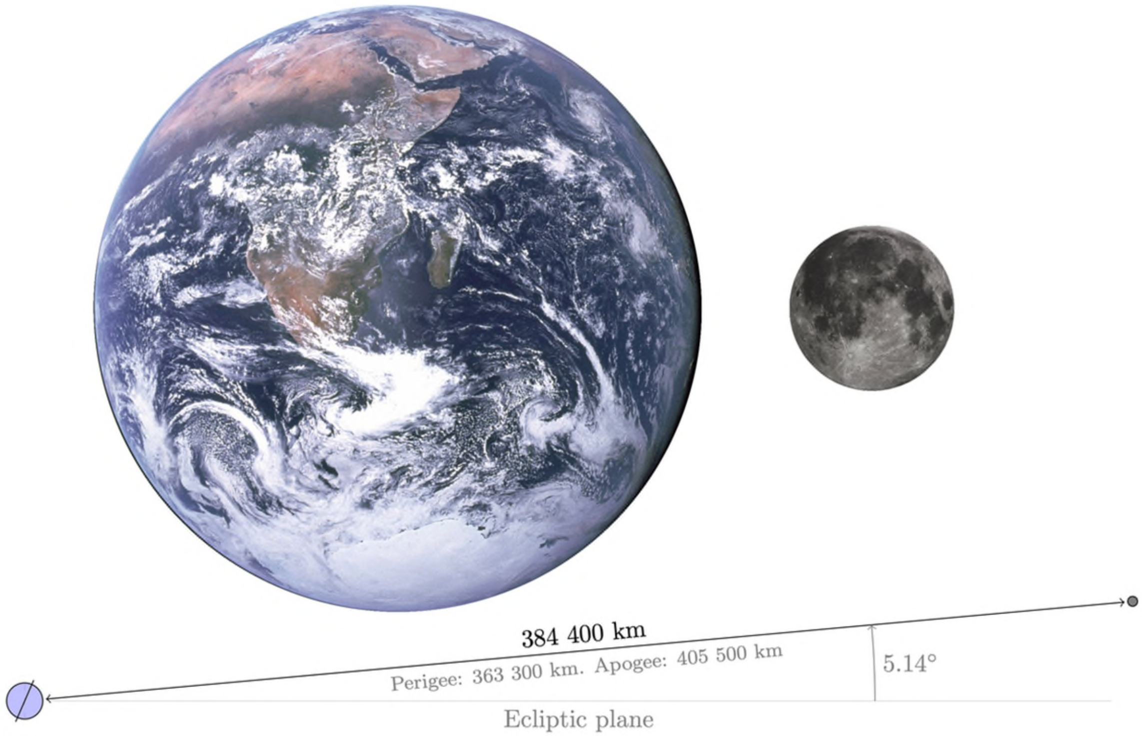
The calculation is now much simpler.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_2 x_2}{m_1 + m_2}$$

$$x_{cm} = \frac{(0.16 \text{ kg}) x_2}{0.64 \text{ kg} + 0.16 \text{ kg}} = \frac{0.16 x_2}{0.80} = \frac{1}{5} x_2$$

Again we get the position to the right of the basketball center at 1/5 the distance between the mass centers. Depending on the separation, the center of mass can be inside the basketball.

At what distance from the center of the Earth is the center of mass for the Earth-Moon system?



Apollo 17 Picture of the Whole Earth: NASA
Telescopic Image of the Full Moon: Gregory H. Revera
Illustration: Vegar Ottesen
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The mass of the Earth is $M = 5.972 \times 10^{24}$ kg and the mass of the Moon is $m = 7.348 \times 10^{22}$ kg. The distance between the Earth and Moon is given in the above figure as $d = 384,400$ km. Choose the center of the Earth as the origin.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{M x_1 + m x_2}{M + m} = \frac{M \cdot 0 + m x_2}{M + m} = \frac{m x_2}{M + m} = \frac{md}{M + m}$$

$$x_{cm} = \frac{md}{M + m} = \frac{7.348 \times 10^{22} \text{ kg} \cdot \text{d}}{5.972 \times 10^{24} \text{ kg} + 7.348 \times 10^{22} \text{ kg}}$$

$$x_{cm} = \frac{7.348 \times 10^{22}}{597.2 \times 10^{22} + 7.348 \times 10^{22}} \cdot \text{d} = \frac{7.348}{597.2 + 7.348} \cdot d$$

$$x_{cm} = \frac{7.348}{597.2 + 7.348} \cdot d = \frac{7.348}{604.548} \cdot d = 0.01215d = (0.01215)(384,400 \text{ km}) = 4670 \text{ km}$$

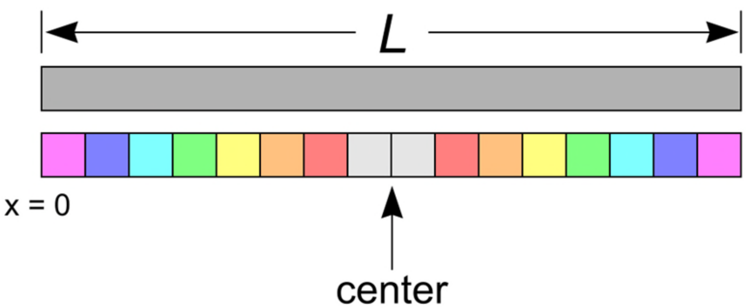
The center of mass is 1700 km below the Earth's surface since the Earth's radius is 6370 km.

I3. The Center of Mass for a Rod of Length L. The usual way to calculate the center of mass for a rod is to use calculus. But just in case you did not get this far yet in Calculus II, I will first do it without calculus. In fact, I will do it several ways, where one method will be using calculus. One of the finest goals in science is to be able to do things more than one way. This aim is also good for anything in life. For example, when I taught at the university, I had many ways to get to work: driving, bumming a ride off a neighbor, my wife taking me, the bus, walking (I did walk home a couple of times), and other ways which I never needed such as a Taxi or Uber.

Feynman once said something to the effect that any good physicist had 6 or 7 ways to understand results. These different approaches include various ways to derive the result and visualizations such as graphs. So here are some different methods to find the *center of mass for a rod*.

Method 1. Symmetry. We have found that for two equal masses, the center of mass is at the midpoint between the two masses.

$$x_{cm} = \frac{mx_1 + mx_2}{m+m} = \frac{x_1 + x_2}{2}$$



So we break up the rod into segments and pair them off, each one on the left with its partner on the right (by color). The center of mass for each pair is at the midpoint between each.

Therefore, the center of mass for the entire rod is at the midpoint of the rod, which midpoint serves as the midpoint for all the mass segment pairs.

$$\boxed{x_{cm} = \frac{L}{2}}$$

Method 2. The Balance Point. The center of mass is the balance point.

Courtesy APN MJM, Wikimedia
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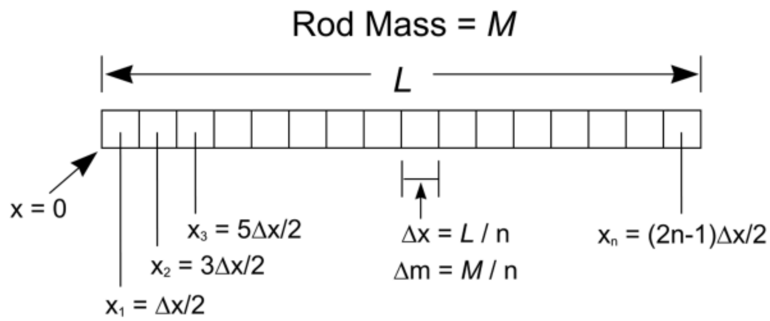
The balance point for a rod is its center, assuming a uniform rod in terms of mass density. Therefore, the center of mass for the rod is $L/2$. For the irregular shape of the bird at the left, the center of mass can be found experimentally by finding the balance point. Since the wings have prominent mass and stick out in the front, it **appears that the bird has more mass on the other side and that it is floating**.

Method 3. The Summing of Small Masses. In the figure below we have broken the rod into little mass segments of Δm. There are n segments; therefore, $\Delta m = \frac{M}{n}$, where M is the total mass of the rod of length L.



The distance from the reference point where $x = 0$ to each of the first three mass segments are

$$x_1 = \frac{\Delta x}{2},\ x_2 = \frac{3\Delta x}{2},\ x_3 = \frac{5\Delta x}{2}.$$

The coefficients in front of the Δx factors are 1, 3, and 5, the first three odd numbers. The n^th odd number can be given as $2n-1$. Pick several counting numbers for n to check this out. We are now ready for the center of mass formula:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + ... + m_n},$$

where $m_1 = m_2 = m_3 = ...\ m_n = \Delta m = \frac{M}{n}$,

and $x_1 = \frac{\Delta x}{2}$, $x_2 = \frac{3\Delta x}{2}$, $x_3 = \frac{5\Delta x}{2}$ ... $x_n = \frac{(2n-1)\Delta x}{2}$.

Using the mass relations, we obtain

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + ... + m_n} = \Delta m \frac{(x_1 + x_2 + x_3 + ...\ x_n)}{M} = \frac{M}{n} \frac{(x_1 + x_2 + x_3 + ...\ x_n)}{M}$$

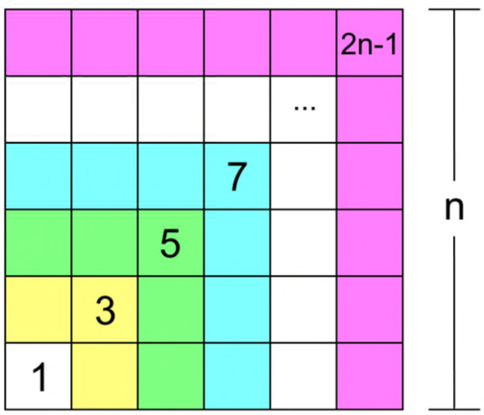
$$x_{cm} = \frac{1}{n}(x_1 + x_2 + x_3 + ...\ x_n)$$

Now substitute in the values for the positions.

$$x_{cm} = \frac{1}{n}\left[\frac{\Delta x}{2} + \frac{3\Delta x}{2} + \frac{5\Delta x}{2} + ... \frac{(2n-1)\Delta x}{2}\right] = x_{cm} = \frac{1}{n}[1 + 3 + 5 + ... + (2n-1)]\frac{\Delta x}{2}$$

Now comes a neat formula for summing odd numbers.

$$1 + 3 + 5 + ... + (2n-1) = n^2$$



1+3+5+7+...+(2n-1) = n²

Here is a visual proof?

You can also verify the first few specific cases.

$$1 = 1^2$$

$$1 + 3 = 4 = 2^2$$

$$1 + 3 + 5 = 9 = 3^2$$

$$1 + 3 + 5 + 7 = 16 = 4^2$$

A physicist is thus happy, but probably not a mathematician. Proof by mathematical induction would please a mathematician. You assume it is true for the n^{th} case and demonstrate that it follows that it is true for the $(n+1)^{th}$ case. Finally, when you show it is true for the 1^{st} case, then it follows that it is true for all n. We will leave mathematical induction for you to try if you want.

Here is where we left off.

$$x_{cm} = \frac{1}{n}\left[1 + 3 + 5 + \ldots + (2n-1)\right]\frac{\Delta x}{2}$$

$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$

Using the formula for the sum of odd numbers,

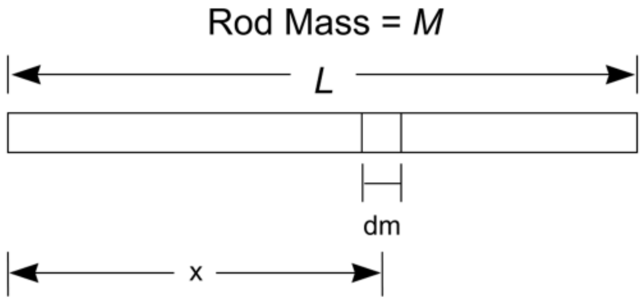
$$x_{cm} = \frac{1}{n}\left[1 + 3 + 5 + \ldots + (2n-1)\right]\frac{\Delta x}{2} = \frac{1}{n}\left[n^2\right]\frac{\Delta x}{2}$$

$$x_{cm} = \frac{n^2}{n}\frac{\Delta x}{2} = n\frac{\Delta x}{2}$$

But $n\Delta x = L$. Therefore,

$$\boxed{x_{cm} = \frac{L}{2}}$$

Method 4. Calculus.



$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \rightarrow \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\frac{M}{L}dx = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{x^2}{2}\bigg|_0^L = \frac{1}{L}\left[\frac{L^2}{2} - \frac{0^2}{2}\right] = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

I4. Conservation of Momentum. We assume in this chapter that we are talking about linear momentum, because there is another type of momentum called angular momentum. We will study angular momentum in another chapter. Start below with our key equation for the center of mass.

$$\overrightarrow{r_{cm}} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \ldots + m_n\vec{r_n}}{m_1 + m_2 + m_3 + \ldots + m_n} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \ldots + m_n\vec{r_n}}{M}$$

The velocity of the center of mass is

$$\overrightarrow{v_{cm}} = \frac{d\overrightarrow{r_{cm}}}{dt} = \frac{1}{M}\frac{d}{dt}(m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \ldots + m_n\vec{r_n}) = \frac{1}{M}(m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \ldots + m_n\vec{v_n})$$

Multiply both sides by M .

$$M\overrightarrow{v_{cm}} = m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \ldots + m_n\vec{v_n}$$

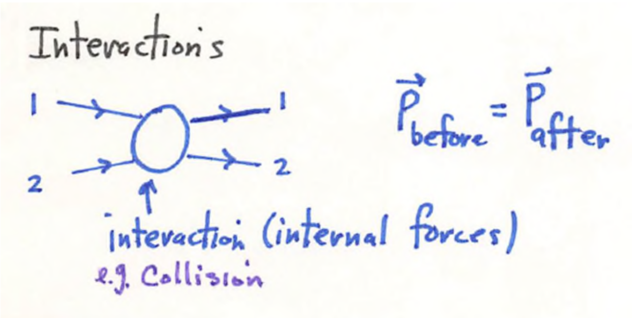
But the right side is the total momentum $\vec{p}$.

Therefore, the total momentum can be expressed as below.

$$\boxed{\vec{p} = M\overrightarrow{v_{cm}}}$$

We see the importance of the center of mass.

If the external forces are zero, then $\overrightarrow{F_{ext}} = \frac{d\vec{p}}{dt} = 0$ and the momentum is a constant.



An example is given here with two particles interacting. The free particles enter an interaction zone where only internal forces occur, such as a collision.

If they interact with each other in this way with no external forces from outside acting on them, the total momentum is a constant. We can then write conservation of momentum as follows.

$$\boxed{\vec{p}_{before} = \vec{p}_{after}}$$

The total momentum for a two-particle system is

$$\vec{p} = m_1\vec{v_1} + m_2\vec{v_2}$$

Let the velocities before the interaction be represented by $\vec{u_1}$ and $\vec{u_2}$; let the velocities after the interaction be represented by $\vec{v_1}$ and $\vec{v_2}$. Remember "u" is before and "v" is after since "u" is before "v" in the alphabet. Conservation of momentum for a two particle system is then given by the formula below.

$$\boxed{m_1\vec{u_1} + m_2\vec{u_2} = m_1\vec{v_1} + m_2\vec{v_2}}$$

Conservation of momentum is a powerful conservation law. If you look at all the particles together in the universe, there are then no external forces because there is no outside! So we can consider conservation of momentum as a general conservation law along with matter-energy. Considering the entire universe, the sum total of matter-energy cannot be created or destroyed. Two Important Conservation Laws, where by energy we include energy due to matter ($E=mc^2$).

1) Conservation of Energy,   2)Conservation of Momentum.

In our course, we do not need to worry about converting energy into matter or vice versa. For our purposes, energy just means energy such as kinetic energy and potential energy. You can add heat energy when friction is involved.

Important Observation. Suppose there are no external forces. Then $\vec{p}_{total} = M\vec{v_{cm}} = \overrightarrow{const}$ and $\vec{p}_{before} = \vec{p}_{after}$. If there is no motion initially, then $\vec{p}_{total} = M\vec{v_{cm}} = \overrightarrow{const} = \vec{0}$, which means $\vec{r_{cm}} = \overrightarrow{const}$, i.e., the center of mass does not change. Also $\vec{p}_{before} = \vec{p}_{after} = \vec{0}$. An example is a bear standing on ice and throwing a ball, which we do in the next section.

I5. Conservation of Momentum and Two Masses.

Problem 1. Figure Skating. This problem is inspired by the photo below. Assume that a 75.0-kg male skater is at rest alone on the ice as a 50.0-kg female skates towards him at a speed of 20.0 km/h. The female skater then jumps on the back of the male skater. What is the speed of the combined two skaters just after the collision? Neglect the external forces of friction, which is justified since the skaters are on ice. Since gravity is not horizontal, it is no worry.

Skating Photo Courtesy Stephen Downes, flickr. License: Attribution-NonCommercial 2.0
Canadian Figure Skating Championship January 21, 2012, Moncton, New Brunswick
Skaters Piper Gilles and Paul Poirier. Moncton Coliseum.

Solution. Use conservation of momentum $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, where $m_1 = 50.0 \text{ kg}$, $u_1 = 20.0 \ \frac{\text{km}}{\text{h}}$, $m_2 = 75.0 \text{ kg}$, $u_2 = 0$, and $v_1 = v_2 = v$. Find v.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \Rightarrow \quad m_1u_1 = (m_1 + m_2)v \quad \Rightarrow \quad (50.0)(20.0) = (50.0 + 75.0)v$$

$$\Rightarrow \quad (1000.0) = (125.0)v \quad \Rightarrow \quad v = 1000.0/125.0 = 8 \quad \Rightarrow \quad \boxed{v = 8\frac{\text{km}}{\text{h}}} = 5\frac{\text{mi}}{\text{h}} = 2.2\frac{\text{m}}{\text{s}} = 7.3\frac{\text{ft}}{\text{s}}$$

Problem 2. Throwing an Object on Ice. A polar bear is throwing a torn ball in the photo below. Assume a young 100.0-kg polar bear is standing on a sheet of ice where friction can be neglected. Suppose a young bear at rest throws a 6-kg bowling ball out in a horizontal direction at 30.0 km/h (19 mi/h). What is the bear's recoil velocity? Note that $\vec{p}_{before} = \vec{p}_{after} = \vec{0}$.

Nanuq Throwing a Ball, Courtesy Tambako The Jaguar, flickr. License: Attribution-NoDerivs 2.0

Use conservation of momentum: $\vec{p}_{before} = \vec{p}_{after}$.

The before momentum $\vec{p}_{before} = 0$ since there are no velocities. Then $\vec{p}_{after} = 0$.

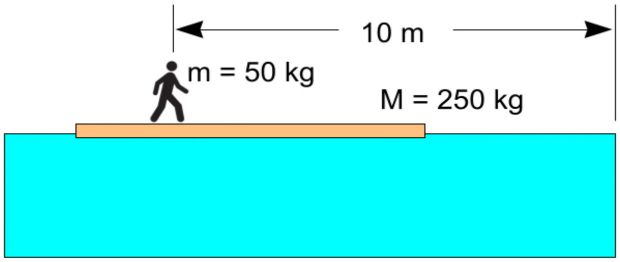
$$0 = m_{ball} v_{ball} + m_{bear} v_{bear} \quad \Rightarrow \quad m_{bear} v_{bear} = -m_{ball} v_{ball} \quad \Rightarrow \quad v_{bear} = -\frac{m_{ball} v_{ball}}{m_{bear}}$$

The minus sign means the bear will move in the opposite direction of the ball.

$$v_{bear} = -\frac{(6 \text{ kg})(30.0 \text{ km/h})}{100.0 \text{ kg}} = -\frac{180}{100}\frac{\text{km}}{\text{h}} = -\frac{18}{10}\frac{\text{km}}{\text{h}} = -\frac{9}{5}\frac{\text{km}}{\text{h}}$$

$\boxed{v_{bear} = 1.8 \frac{\text{km}}{\text{h}}}$ and to the right in the figure. This speed equals 1.1 mi/h = 50 cm/s.

I6. Walking on a Boat. The walking on a raft or boat problem is a classic problem involving conservation of momentum and analyzing center of mass.



Problem. A 50-kg person initially at rest begins walking on an initially stationary 250-kg raft. The raft begins to drift backwards in response to the walking. The small amount of friction between the raft and the water can be neglected.

(i) Find the velocity v_{raft} of the raft if the person starts to walk with speed $v_{rel} = 1.2 \frac{\text{m}}{\text{s}}$ relative to the raft.

(ii) If the person starts out a distance D_{before} = 10 m from land, how far D_{after} is the person from land after walking towards the land a distance $d_{rel} = 2.4 \text{ m}$ relative to the raft?

Credit Line: Clipart person in above figure is from a Public Domain US National Park Service sign.

Solution. There are no external forces in the horizontal direction. Therefore, momentum is conserved. Note that the total momentum is zero throughout since initially the person and the raft are at rest.

(i) Raft Velocity. Conservation of momentum: $p_{before} = p_{after}$, where $p_{before} = 0$ and $p_{after} = 0$.

$$p_{before} = p_{after} \quad => \quad 0 + 0 = mv_{person} + Mv_{raft}$$

We have to be careful since as the person walks relative to the raft, the raft is recoiling backwards. The velocity of the person relative to the ground is therefore

$v_{person} = v_{rel} + v_{raft}$, where $v_{raft} < 0$, meaning the raft velocity is to the left.

$$0 + 0 = mv_{person} + Mv_{raft} \quad => \quad 0 = m(v_{rel} + v_{raft}) + Mv_{raft}$$

$$0 = mv_{rel} + mv_{raft} + Mv_{raft} \quad => \quad 0 = mv_{rel} + (m + M)v_{raft} \quad => \quad (m + M)v_{raft} = -mv_{rel}$$

$$\boxed{v_{raft} = -(\frac{m}{m + M})v_{rel}} \quad => \quad v_{raft} = -(\frac{50 \text{ kg}}{50 \text{ kg} + 250 \text{ kg}})(1.2 \frac{\text{m}}{\text{s}}) = -\frac{50}{300}(1.2 \frac{\text{m}}{\text{s}}) = -\frac{1}{6}(1.2 \frac{\text{m}}{\text{s}})$$

$$\boxed{v_{raft} = -0.2 \frac{\text{m}}{\text{s}}} \qquad \boxed{v_{person} = v_{rel} + v_{raft} = (1.2 - 0.2)\frac{\text{m}}{\text{s}} = 1.0 \frac{\text{m}}{\text{s}}}$$

(ii) What is the distance from the person to land after walking, i.e., what is D_{after}?

Method 1. Conservation of Momentum. We solve the problem like we did above, obtaining

$$\boxed{v_{raft} = -(\frac{m}{m+M})v_{rel}}$$

Now we use the fact that $v = \frac{\Delta x}{\Delta t}$ and write

$$\frac{\Delta x_{raft}}{\Delta t} = -(\frac{m}{m+M})\frac{\Delta x_{rel}}{\Delta t} .$$

The Δt components drop out.

$$\Delta x_{raft} = -(\frac{m}{m+M})\Delta x_{rel} \quad => \quad \Delta x_{raft} = -(\frac{m}{m+M})d_{rel}$$

$$\Delta x_{raft} = -(\frac{50}{50+250})(2.4) = -(\frac{50}{300})(2.4) = -\frac{1}{6}(2.4) = -0.4 \text{ m}$$

The raft moves to the left subtracting from the distance walked on the raft:

$$2.4 \text{ m} - 0.4 \text{ m} = 2.0 \text{ m}$$

The person moves 2.0 m to the land.

$$D_{after} = D_{before} - 2.0 \text{ m}$$

$$D_{after} = 10 \text{ m} - 2 \text{ m} = 8 \text{ m}$$

$$\boxed{D_{after} = 8 \text{ m}}$$

Method 2. Center of Mass. Remember from the last section, when there are no external forces and we have a case where $\vec{p}_{total} = M\overrightarrow{v_{cm}} = \overrightarrow{const} = \vec{0}$. Then since $\overrightarrow{v_{cm}} = \frac{d\overrightarrow{r_{cm}}}{dt} = 0$, the center of mass point is fixed: $\overrightarrow{r_{cm}} = const$. We have a one-dimension problem with

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = const .$$

Therefore,

$$\Delta x_{cm} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{m_1 + m_2} = 0$$

$$m_1 \Delta x_1 + m_2 \Delta x_2 = 0$$

$$m_{person}(d_{rel} + \Delta x_{raft}) + m_{raft} \Delta x_{raft} = 0$$

$$m_{person} d_{rel} + m_{person} \Delta x_{raft} + m_{raft} \Delta x_{raft} = 0$$

$$m_{person} d_{rel} + (m_{person} + m_{raft}) \Delta x_{raft} = 0$$

$$(m_{person} + m_{raft}) \Delta x_{raft} = -m_{person} d_{rel}$$

$$\Delta x_{raft} = -\frac{m_{person}}{m_{person} + m_{raft}} d_{rel}$$

$$\Delta x_{raft} = -\frac{50}{50 + 250}(2.4) = -\frac{50}{300}(2.4) = -\frac{1}{6}(2.4) = -0.4 \text{ m}$$

The person moves 2.4 m – 0.4 m = 2.0 m towards the land.

The person is finally 10 m – 2 m = 8 m from the land.

We get the same answer as we did for Method 1.

I7. Impulse. Newton's Second Law states that the net applied force is equal to the change in momentum.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Looking at linear situations, we can write

$$F = \frac{dp}{dt}$$

Now we consider a situation where we observe a change of momentum Δp over a finite time interval Δt. In this case, we can express the average force $\overline{F}$ as

$$\overline{F} = \frac{\Delta p}{\Delta t}.$$

There is nothing really tricky here. It is similar to an average velocity when looking at an overall distance and time:

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

I used to drive from Asheville to Winston-Salem regularly to take my kids to a piano teacher. The distance from Asheville to Winston-Salem is about 240 km (150 mi). We would stop at least once along the way so it took about 3 hours. The average velocity was

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{240\text{ km}}{3\text{ h}} = 80\frac{\text{km}}{\text{h}}\text{, which is } 50\frac{\text{mi}}{\text{h}}.$$

The impulse is defined as

$$\Delta p = \overline{F}\Delta t.$$

Courtesy ataelw, flickr. License: Attribution 2.0.
2011 WTA Rogers Cup, Toronto

However, there is one important distinction: the time interval must be very small. That's why we call it "IMPULSE."

The short time interval when a tennis racket hits the ball is a good example. The letter J is often used.

$$\boxed{J = \overline{F}\Delta t = \Delta p}$$

Application: Collision and Safety.

Courtesy perthhdproductions, flickr. License: Attribution 2.0.
Car Crash, Karrinyup Road, Stirling, Western Australia (WA), taken March 27, 2012.

We will show how impulse is important in analyzing collisions. First, what is the data easily obtained for a collision? It is not the time duration of the collision. That happens very fast. The easiest observations are the speed and the distance traveled during the collision. We can get the speed either from the driver or making a good estimate given the road and witnesses. Let's take a speed of 40.0 km/h (25 mi/h). The distance traveled during impact can be determined from the deformation of the car. We will take this distance to be 40.0 cm (16 inches). Now we go to work.

We pull out the kinematics formula that has the speeds and distance traveled so we can arrive at an average acceleration during the collision.

$$2\bar{a}d = v^2 - v_0^2$$

The bar over the acceleration reminds us that we are calculating an average acceleration in this case. The initial speed is $v_0 = 40.0\ \frac{\text{km}}{\text{h}}$, and the distance $d = 0.40\ \text{m}$. It will be convenient to have the speed in meters per second.

$$v_0 = 40.0\ \frac{\text{km}}{\text{h}} \cdot \frac{1000\ \text{m}}{1\ \text{km}} \cdot \frac{1\ \text{h}}{3600\ \text{s}} = 40 \cdot \frac{10}{36}\frac{\text{m}}{\text{s}} = 40 \cdot \frac{5}{18}\frac{\text{m}}{\text{s}} = 11.11\frac{\text{m}}{\text{s}}$$

Then, the average acceleration can be determined from $2\bar{a}d = v^2 - v_0^2$ with $d = 0.40 \text{ m}$, $v = 0$ as the car is at rest at the end, and $v_0 = 11.11 \dfrac{\text{m}}{\text{s}}$.

$$2\bar{a}(0.40 \text{ m}) = 0^2 - (11.11 \frac{\text{m}}{\text{s}})^2$$

Note how keeping the dimensions would alert us if we forgot to convert the speed to m/s.

$$\bar{a}(0.80 \text{ m}) = -123.4 \frac{\text{m}^2}{\text{s}^2}$$

$$\bar{a} = -\frac{123.4}{0.80 \text{ m}} \frac{\text{m}^2}{\text{s}^2} = -154.3 \frac{\text{m}}{\text{s}^2}$$

The negative sign indicates deceleration.

The g-force is

$$\bar{a} = -154.3 \frac{\text{m}}{\text{s}^2} \cdot \frac{g}{9.8 \frac{\text{m}}{\text{s}^2}} = -15.7g = -16g$$

Typically the absolute magnitude is reported, i.e., the 16 g without the minus sign.

This g-force produces a lot of force on your body!

Elements that go into injury probabilities include the direction of the force and length of time. Below is a short table listing some g-forces taken from a big table on Wikipedia.

Example	g-force
Gravitron Amusement Ride	2.5 – 3 g
Uninhibited Sneezing after Sniffing Ground Pepper	2.9 g
Space Shuttle (Launch and Reentry)	3 g
High-g Roller Coasters	3.5 – 6.3 g
Hearty Greeting Slap on Upper Back	4.1 g
Top Fuel Drag Racing World Record of 4.4 s over 1/4 mi	4.2 g
Formula One Racing Car Under Heavy Braking	6.3 g
Apollo 16 on Reentry	7.19 g
Maximum g-force Permitted in Russian Mikoyan MiG-35	10 g

Data from a Longer Table at Wikipedia

We can find the time of the collision from the kinematic formula that has the velocities, the distance, and the time. That equation is

$$x = x_0 + \frac{1}{2}(v_0 + v)t \ .$$

Entering the data,

$$0.40 \text{ m} = 0 + \frac{1}{2}(11.11\frac{\text{m}}{\text{s}} + 0)t$$

$$0.80 \text{ m} = 11.11\frac{\text{m}}{\text{s}} \cdot t \quad \Rightarrow \quad t = \frac{0.80 \text{ m}}{11.11\frac{\text{m}}{\text{s}}} \quad \Rightarrow \quad t = 0.072 \text{ s}$$

$$t = 72 \text{ milliseconds}$$

$$\boxed{t = 72 \text{ ms}}$$

At this point, the impulse concept is important for our analysis: $J = \overline{F}\Delta t = \Delta p$. We know Δp . However, an airbag can increase the deceleration time for the driver. Consider Δp as a constant for the collision. Increasing Δt for the driver, decreases the g-force for the driver. The product is the impulse, which remains a constant for the analysis.

$$J = \overline{F}\Delta t = \Delta p = const$$

$$\overline{F} = \frac{const}{\Delta t} \quad \Rightarrow \quad \text{g-force} = \frac{\overline{F}}{m} = \frac{const}{\Delta t} \quad \Rightarrow \quad \text{g-force} = \frac{const}{\Delta t}$$

To apply this formula to our problem, we have a g-force of 16 g for 72 ms. Therefore,

$$\text{g-force} = \frac{16g \cdot 72}{\Delta t(\text{in ms})},$$

since when $\Delta t = 72 \text{ ms}$, you get your expected 16 g for the g-force.

We remind ourselves in parentheses that the time interval must be in ms.

Below is a table listing some pairs of values for Δt and $\overline{F}$ using

$$\text{g-force} = \frac{16g \cdot 72}{\Delta t(\text{in ms})} = \frac{1152g}{\Delta t(\text{in ms})} \ .$$

The table uses $\text{g-force} = \dfrac{1152g}{\Delta t(\text{in ms})}$ to calculate the values. The longer it takes the driver to come to a complete stop due to the airbag, the lower the average g-force.

Δt	Average Force g-force
72 ms	16 g
80 ms	14.4 g
90 ms	12.8 g
100 ms	11.5 g
150 ms	7.68 g
200 ms	5.8 g

You can see that increasing the time interval for the driver to come to a complete stop, can significantly decrease the average g-force.

In a collision, it takes the car sensor about 15 ms to decide whether to deploy the airbag or not. Once the decision to deploy is made, it can take in additional 20 ms to inflate, i.e., 35 ms after impact inflation is completed. By 60 ms, the driver is in contact with the airbag and is pushing on the airbag. This contact can last about 35 to 40 ms. We make a running table below of these times, which come from the website tristanmac.tripod.com/id8.html.

Time from Impact	Description of Event
0 ms	Car Begins its Contact with Tree
15 ms	Sensor Decides to Deploy Airbag
35 ms	Airbag is Finished Inflating
60 ms	Driver in Contact with Airbag
100 ms	Driver Fully Comes to Rest

Estimate of Airbag Interaction with Driver.
Data from tristanmac.tripod.com/id8.html.

Consulting the g-force table where $\Delta t = 100 \text{ ms}$, the g-force is 11.5 g.

Since the time is so short, 100 ms, which is 1/10 of a second, the driver will experience an average g-force of 10 g for only 1/10 of a second. There is a good chance that a healthy sturdy person sitting in a secure position with a seat belt, harness, and airbag may be okay.

Finally, we make an analogy with energy. The second pair below are written in calculus, allowing for a variable force. Think of those as areas under the F vs x graph and F vs t graph.

$$W = Fx = \Delta K \qquad J = Ft = \Delta p$$

$$\boxed{W = \int_{x_1}^{x_2} F dx = \Delta K = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2} \qquad \boxed{J = \int_{t_1}^{t_2} F dt = \Delta p = m v_2 - m v_1}$$